Generalization of Wrench's identity for Lambert series

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April 6, 2014

Abstract

In this short note we give a generalization of an identity for Lambert series by Wrench.

1 Wrench's identity

The following relation is given by Knuth (attributed to J. R. Wrench, Jr.) [3, p.644, solution to exercise 5.2.3-27]:

$$W(q) := \sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = \sum_{n\geq 1} q^{n^2} \left[a_n + \sum_{k\geq 1} (a_n + a_{n+k})\, q^{k\,n} \right] \tag{1}$$

To verify this identity, write the Lambert sum as

$$W(q) = \sum_{n\geq 1} \sum_{k\geq 1} a_n\, q^{k\,n} \tag{2}$$

and write the summands in a table (cf. [4]) as shown in figure 1. Then take sums starting from the diagonal entries $a_n\, q^{n^2}$, taking both the terms to the right in the same row $\sum_{k\geq 1} a_n\, q^{n^2+k\,n}$, and the terms below in the same column $\sum_{j\geq 1} a_{n+j}\, q^{n^2+j\,n}$. To obtain (1), combine into a single sum, replacing j by k in the column sums.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$\cdots$	k	$\cdots$
$n=1$	$a_1\,q^1$	$a_1\,q^2$	$a_1\,q^3$	$a_1\,q^4$	$a_1\,q^5$	$\cdots$	$a_1\,q^k$	$\cdots$
$n=2$	$a_2\,q^2$	$a_2\,q^4$	$a_2\,q^6$	$a_2\,q^8$	$a_2\,q^{10}$	$\cdots$	$a_2\,q^{2k}$	$\cdots$
$n=3$	$a_3\,q^3$	$a_3\,q^6$	$a_3\,q^9$	$a_3\,q^{12}$	$a_3\,q^{15}$	$\cdots$	$a_3\,q^{3k}$	$\cdots$
$n=4$	$a_4\,q^4$	$a_4\,q^8$	$a_4\,q^{12}$	$a_4\,q^{16}$	$a_4\,q^{20}$	$\cdots$	$a_4\,q^{4k}$	$\cdots$
$n=5$	$a_5\,q^5$	$a_5\,q^{10}$	$a_5\,q^{15}$	$a_5\,q^{20}$	$a_5\,q^{25}$	$\cdots$	$a_5\,q^{5k}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
n	$a_n\,q^n$	$a_n\,q^{2n}$	$a_n\,q^{3n}$	$a_n\,q^{4n}$	$a_n\,q^{5n}$	$\cdots$	$a_n\,q^{n\,k}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

Figure 1: The double sum in relation (2) written as a table.

2 The generalization

We first derive

$$\sum_{n\geq 1} \frac{a_n\,q^n}{1-b_n\,q^n} = \sum_{n\geq 1} q^{n^2} \left[a_n\,b_n^{n-1} + \sum_{k\geq 1} \left(a_n\,b_n^{n+k-1} + a_{n+k}\,b_{n+k}^{n-1} \right)\,q^{n\,k} \right] \tag{3}$$

As before, write as a double sum

$$\sum_{n\geq 1} \sum_{k\geq 1} a_n\,b_n^{k-1}\,q^{k\,n} \tag{4}$$

Take sums starting from the diagonal entries $a_n\,b_n^{n-1}\,q^{n^2}$, the terms to the right give $\sum_{k\geq 1} a_n\,b_n^{n+k-1}\,q^{n^2+k\,n}$, the terms below give $\sum_{j\geq 1} a_{n+j}\,b_{n+j}^{n-1}\,q^{n^2+j\,n}$. Write as a single sum to obtain (3).

Replacing a_n by $a_n\,t^n/q^n$ and simplifying gives the desired identity:

$$\sum_{n\geq 1} \frac{a_n\,t^n}{1-b_n\,q^n} =$$

$$= \sum_{n\geq 1} q^{n\,(n-1)}\,t^n \left[a_n\,b_n^{n-1} + \sum_{k\geq 1} \left(a_n\,b_n^{n+k-1}\,q^k + a_{n+k}\,b_{n+k}^{n-1}\,t^k \right)\,q^{k(n-1)} \right] \tag{5}$$

For the special case $a_n = 1/t$ and $b_n = x/q$ there is Agarwal's relation, see [2] (note the summation starts with $n=0$),

$$\sum_{n\geq 0} \frac{t^n}{1-x\,q^n} = \sum_{n\geq 0} \frac{(1-x\,t\,q^{2n})}{(1-x\,q^n)\,(1-t\,q^n)}\,(x\,t)^n\,q^{n^2}$$

and also, see [1],

$$\sum_{n\geq 0} \frac{t^n}{1-x\,q^n} = \sum_{n\geq 0} \frac{(q;q)_n}{(x;q)_{n+1}\,(t;q)_{n+1}}\,(x\,t)^n\,q^{(n^2-n)/2}$$

where $(z;q)_n = (1-z)\,(1-z\,q)\,(1-z\,q^2)\;\cdots\;(1-z\,q^{n-1})$ and $(z;q)_0 = 1$. Writing identity (5) in a similar way doesn't seem easily possible.

References

[1] Jörg Arndt: On computing the generalized Lambert series, arXiv:1202.6525 [math.CA] (2012).

[2] R. P. Agarwal: Lambert series and Ramanujan, Proceedings of the Indian Academy of Science (Mathematical Sciences), vol.103, no.3, pp.269-293, (December-1993).

[3] Donald E. Knuth: The Art of Computer Programming, second edition, Volume 3: Sorting and Searching, Addison-Wesley, (1997).

[4] Thomas J. Osler, Abdul Hassen: On generalizations of Lambert's series, International Journal of Pure and Applied Mathematics, vol.43, no.4, pp.465-484, (2008).