

Subset-lex: did we miss an order?

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Abstract

We generalize a well-known algorithm for the generation of all subsets of a set in lexicographic order with respect to the sets as lists of elements (subset-lex order). We obtain algorithms for various combinatorial objects such as the subsets of a multiset, compositions and partitions represented as lists of parts, and for certain restricted growth strings. The algorithms are often loopless and require at most one extra variable for the computation of the next object. The performance of the algorithms is very competitive even when not loopless. A Gray code corresponding to the subset-lex order and a Gray code for compositions that was found during this work are described.

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0:	[.]	{ }	[.]	{ }
1:	[1]	{ 0 }	[. . . . 1]	{ 4 }
2:	[1 1 . . .]	{ 0, 1 }	[. . . . 1 1]	{ 3 }
3:	[1 1 1 . .]	{ 0, 1, 2 }	[. . . . 1 1 1]	{ 3, 4 }
4:	[1 1 1 1 .]	{ 0, 1, 2, 3 }	[. . 1 . .]	{ 2 }
5:	[1 1 1 1 1]	{ 0, 1, 2, 3, 4 }	[. . 1 . 1]	{ 2, 4 }
6:	[1 1 1 . 1]	{ 0, 1, 2, 4 }	[. . 1 1 .]	{ 2, 3 }
7:	[1 1 . 1 .]	{ 0, 1, 3 }	[. . 1 1 1]	{ 2, 3, 4 }
8:	[1 1 . 1 1]	{ 0, 1, 3, 4 }	[. 1 . . .]	{ 1 }
9:	[1 1 . . 1]	{ 0, 1, 4 }	[. 1 . . 1]	{ 1, 4 }
10:	[1 . 1 . .]	{ 0, 2 }	[. 1 . 1 .]	{ 1, 3 }
11:	[1 . 1 1 .]	{ 0, 2, 3 }	[. 1 . 1 1]	{ 1, 3, 4 }
12:	[1 . 1 1 1]	{ 0, 2, 3, 4 }	[. 1 1 . .]	{ 1, 2 }
13:	[1 . 1 . 1]	{ 0, 2, 4 }	[. 1 1 . 1]	{ 1, 2, 4 }
14:	[1 . . 1 .]	{ 0, 3 }	[. 1 1 1 .]	{ 1, 2, 3 }
15:	[1 . . 1 1]	{ 0, 3, 4 }	[. 1 1 1 1]	{ 1, 2, 3, 4 }
16:	[1 . . . 1]	{ 0, 4 }	[1]	{ 0 }
17:	[. 1 . . .]	{ 1 }	[1 . . . 1]	{ 0, 4 }
18:	[. 1 1 . .]	{ 1, 2 }	[1 . . 1 .]	{ 0, 3 }
19:	[. 1 1 1 .]	{ 1, 2, 3 }	[1 . . 1 1]	{ 0, 3, 4 }
20:	[. 1 1 1 1]	{ 1, 2, 3, 4 }	[1 . 1 . .]	{ 0, 2 }
21:	[. 1 1 . 1]	{ 1, 2, 4 }	[1 . 1 . 1]	{ 0, 2, 4 }
22:	[. 1 . 1 .]	{ 1, 3 }	[1 . 1 1 .]	{ 0, 2, 3 }
23:	[. 1 . 1 1]	{ 1, 3, 4 }	[1 . 1 1 1]	{ 0, 2, 3, 4 }
24:	[. 1 . . 1]	{ 1, 4 }	[1 1 . . .]	{ 0, 1 }
25:	[. . 1 . .]	{ 2 }	[1 1 . . 1]	{ 0, 1, 4 }
26:	[. . 1 1 .]	{ 2, 3 }	[1 1 . 1 .]	{ 0, 1, 3 }
27:	[. . 1 1 1]	{ 2, 3, 4 }	[1 1 . 1 1]	{ 0, 1, 3, 4 }
28:	[. . . 1 .]	{ 2, 4 }	[1 1 1 . .]	{ 0, 1, 2 }
29:	[. . . . 1]	{ 3 }	[1 1 1 . 1]	{ 0, 1, 2, 4 }
30:	[. . . . 1 1]	{ 3, 4 }	[1 1 1 1 .]	{ 0, 1, 2, 3 }
31:	[. . . . 1]	{ 4 }	[1 1 1 1 1]	{ 0, 1, 2, 3, 4 }

Figure 1: Two lexicographic orders for the subsets of a 5-element set: ordering using the sets as lists of elements (left) and ordering using the characteristic words (right). Dots are used to denote zeros in the characteristic words.

1 Two lexicographic orders for binary words

Two simple ways to represent subsets of a set $\{0, 1, 2, \dots, n - 1\}$ are as lists of elements and as the characteristic word, the binary word with ones at the positions corresponding to the elements in the subset. Sorting all subsets by list of characteristic words gives the right columns of figure 1, sorting by the list of elements gives the left columns. We will concern ourselves with the generalization of the ordering by the lists which we call *subset-lex* order.

The algorithms for computing the successor or predecessor of a (nonempty) subset S in subset-lex order are well-known¹. In the following let z be the last, and y the next to last element in S . We call an element minimal or maximal if it is respectively the smallest or greatest element in the superset

Algorithm 1 (Next-SL2). *Compute the successor of the subset S .*

1. If there is just one element, and it is maximal, stop.
2. If z is not maximal, append $z + 1$.
3. Otherwise remove z and y , then append $y + 1$.

Algorithm 2 (Prev-SL2). *Compute the predecessor of the subset S .*

1. If there is just one element, and it is minimal, stop.

¹For example, an algorithm for the k -subsets of an n -set is given in [25, Algorithm LEXSUB, p.18], see section 11 for an implementation.

2. If $z - 1 \in S$, remove z .
3. Otherwise remove z , append $z - 1$ and the maximal element.

C++ implementations of all algorithms discussed here are given in the FXT library [1]. Here and elsewhere we give slightly simplified versions of the actual code, like omitting modifiers such as `public`, `protected`, and `private`. The type `ulong` is short for unsigned long.

```

1 // FILE: src/comb/subset-lex.h
2 class subset_lex
3 // Nonempty subsets of the set {0,1,2,...,n-1} in subset-lex order.
4 // Representation as list of parts.
5 // Loopless generation.
6 {
7     ulong n_; // number of elements in set, should have n>=1
8     ulong k_; // index of last element in subset
9     ulong n1_; // == n - 1 for n >=1, and == 0 for n==0
10    ulong *x_; // x[0...k-1]: subset of {0,1,2,...,n-1}
11
12    subset_lex(ulong n)
13    // Should have n>=1,
14    // for n==0 one set with the element zero is generated.
15    {
16        n_ = n;
17        n1_ = (n_ ? n_ - 1 : 0 );
18        x_ = new ulong[n_ + (n_==0)];
19        first();
20    }

```

We will always use the method names `first()` and `last()` for the first and last element in the respective list of combinatorial objects.

```

1     ulong first()
2     {
3         k_ = 0;
4         x_[0] = 0;
5         return k_ + 1;
6     }
7
8     ulong last()
9     {
10        k_ = 0;
11        x_[0] = n1_;
12        return k_ + 1;
13    }

```

The methods for computing the successor and predecessor will always be called `next()` and `prev()` respectively.

```

1     ulong next()
2     // Return number of elements in subset.
3     // Return zero if current is last.
4     {
5         if ( x_[k_] == n1_ ) // last element is max?
6         {
7             if ( k_==0 ) return 0;
8
9             --k_; // remove last element
10            x_[k_] += 1; // increment last element
11        }
12        else // add next element from set:
13        {
14            ++k_;
15            x_[k_] = x_[k_-1] + 1;
16        }
17        return k_ + 1;
18    }
19
20
21
22    ulong prev()
23    // Return number of elements in subset.
24    // Return zero if current is first.
25    {

```

```

26     if ( k_ == 0 ) // only one element ?
27     {
28         if ( x_[0]==0 ) return 0;
29
30         x_[0] -= 1; // decrement last (and only) element
31         ++k_;
32         x_[k_] = n1_; // append maximal element
33     }
34     else
35     {
36         if ( x_[k_] == x_[k-1] + 1 ) --k_; // remove last element
37         else
38         {
39             x_[k_] -= 1; // decrement last element
40             ++k_;
41             x_[k_] = n1_; // append maximal element
42         }
43     }
44     return k_ + 1;
45 }
46

```

A program demonstrating usage of the class is

```

1 // FILE: demo/comb/subset-lex-demo.cc
2     ulong n = 6;
3     subset_lex S(n);
4     do
5     {
6         // visit subset
7     }
8     while ( S.next() );

```

2 Subset-lex order for multisets

For the internal representation of the subsets of a multiset, we could use lists of pairs (e, m) where e is the element and m its multiplicity. This choice would lead to loopless algorithms as will become clear in a moment. The disadvantage of this representation, however, is that the generalizations we will find would have a slightly more complicated form. We will instead use generalized characteristic words, where nonzero entries can be at most the multiplicity of the element in question. For a multiset $\{0^{m(0)}, 1^{m(1)}, \dots, k^{m(k)}\}$ these are the mixed radix numbers with radix vector $[m(0)+1, m(1)+1, \dots, m(k)+1]$. The subsets of the set $\{0^1, 1^2, 2^2, 3^3\}$ in subset-lex order are shown in figure 2.

The algorithms for computing the successor or predecessor of a (nonempty) subset S of a multiset in subset-lex are now given. In the following let z be the last element in S .

Algorithm 3 (Next-SL). *Compute the successor of the subset S .*

1. If the multiplicity of z is not maximal, increase it; return.
2. If z is not maximal, append $z + 1$ with multiplicity 1; return.
3. If z is the only element, stop.
4. Remove z and decrease the multiplicity of next to last nonzero element y , then append $y + 1$ with multiplicity 1.

The description omits the case of the empty set, the implementation takes care of this case by initially pointing to the leftmost digit of the characteristic word, which is zero. With the first call to the routine it is changed to 1.

Algorithm 4 (Prev-SL). *Compute the predecessor of the subset S .*

0:	[. . . .]	{ }
1:	[1 . . .]	{ 0 }
2:	[1 1 . .]	{ 0, 1 }
3:	[1 2 . .]	{ 0, 1, 1 }
4:	[1 2 1 .]	{ 0, 1, 1, 2 }
5:	[1 2 2 .]	{ 0, 1, 1, 2, 2 }
6:	[1 2 2 1]	{ 0, 1, 1, 2, 2, 3 }
7:	[1 2 2 2]	{ 0, 1, 1, 2, 2, 3, 3 }
8:	[1 2 2 3]	{ 0, 1, 1, 2, 2, 3, 3, 3 }
9:	[1 2 1 1]	{ 0, 1, 1, 2, 3 }
10:	[1 2 1 2]	{ 0, 1, 1, 2, 3, 3 }
11:	[1 2 1 3]	{ 0, 1, 1, 2, 3, 3, 3 }
12:	[1 2 . 1]	{ 0, 1, 1, 3 }
13:	[1 2 . 2]	{ 0, 1, 1, 3, 3 }
14:	[1 2 . 3]	{ 0, 1, 1, 3, 3, 3 }
15:	[1 1 1 .]	{ 0, 1, 2 }
16:	[1 1 2 .]	{ 0, 1, 2, 2 }
17:	[1 1 2 1]	{ 0, 1, 2, 2, 3 }
18:	[1 1 2 2]	{ 0, 1, 2, 2, 3, 3 }
19:	[1 1 2 3]	{ 0, 1, 2, 2, 3, 3, 3 }
20:	[1 1 1 1]	{ 0, 1, 2, 3 }
21:	[1 1 1 2]	{ 0, 1, 2, 3, 3 }
22:	[1 1 1 3]	{ 0, 1, 2, 3, 3, 3 }
23:	[1 1 . 1]	{ 0, 1, 3 }
24:	[1 1 . 2]	{ 0, 1, 3, 3 }
25:	[1 1 . 3]	{ 0, 1, 3, 3, 3 }
26:	[1 . 1 .]	{ 0, 2 }
27:	[1 . 2 .]	{ 0, 2, 2 }
28:	[1 . 2 1]	{ 0, 2, 2, 3 }
29:	[1 . 2 2]	{ 0, 2, 2, 3, 3 }
30:	[1 . 2 3]	{ 0, 2, 2, 3, 3, 3 }
31:	[1 . 1 1]	{ 0, 2, 3 }
32:	[1 . 1 2]	{ 0, 2, 3, 3 }
33:	[1 . 1 3]	{ 0, 2, 3, 3, 3 }
34:	[1 . . 1]	{ 0, 3 }
35:	[1 . . 2]	{ 0, 3, 3 }
36:	[1 . . 3]	{ 0, 3, 3, 3 }
37:	[. 1 . .]	{ 1 }
38:	[. 2 . .]	{ 1, 1 }
39:	[. 2 1 .]	{ 1, 1, 2 }
40:	[. 2 2 .]	{ 1, 1, 2, 2 }
41:	[. 2 2 1]	{ 1, 1, 2, 2, 3 }
42:	[. 2 2 2]	{ 1, 1, 2, 2, 3, 3 }
43:	[. 2 2 3]	{ 1, 1, 2, 2, 3, 3, 3 }
44:	[. 2 1 1]	{ 1, 1, 2, 3 }
45:	[. 2 1 2]	{ 1, 1, 2, 3, 3 }
46:	[. 2 1 3]	{ 1, 1, 2, 3, 3, 3 }
47:	[. 2 . 1]	{ 1, 1, 3 }
48:	[. 2 . 2]	{ 1, 1, 3, 3 }
49:	[. 2 . 3]	{ 1, 1, 3, 3, 3 }
50:	[. 1 1 .]	{ 1, 2 }
51:	[. 1 2 .]	{ 1, 2, 2 }
52:	[. 1 2 1]	{ 1, 2, 2, 3 }
53:	[. 1 2 2]	{ 1, 2, 2, 3, 3 }
54:	[. 1 2 3]	{ 1, 2, 2, 3, 3, 3 }
55:	[. 1 1 1]	{ 1, 2, 3 }
56:	[. 1 1 2]	{ 1, 2, 3, 3 }
57:	[. 1 1 3]	{ 1, 2, 3, 3, 3 }
58:	[. 1 . 1]	{ 1, 3 }
59:	[. 1 . 2]	{ 1, 3, 3 }
60:	[. 1 . 3]	{ 1, 3, 3, 3 }
61:	[. . 1 .]	{ 2 }
62:	[. . 2 .]	{ 2, 2 }
63:	[. . 2 1]	{ 2, 2, 3 }
64:	[. . 2 2]	{ 2, 2, 3, 3 }
65:	[. . 2 3]	{ 2, 2, 3, 3, 3 }
66:	[. . 1 1]	{ 2, 3 }
67:	[. . 1 2]	{ 2, 3, 3 }
68:	[. . 1 3]	{ 2, 3, 3, 3 }
69:	[. . . 1]	{ 3 }
70:	[. . . 2]	{ 3, 3 }
71:	[. . . 3]	{ 3, 3, 3 }

Figure 2: Subsets of the multiset $\{0^1, 1^2, 2^2, 3^3\}$ in subset-lex order. Dots are used to denote zeros in the (generalized) characteristic words.

1. If the set is empty, stop.
2. Decrease the multiplicity of z .
3. If the new multiplicity is nonzero, return.
4. Increase the multiplicity of $z - 1$ and append the maximal element.

Clearly both algorithms are loopless for the representation using a list of pairs. With the representation as characteristic words algorithm 3 involves a loop in the last step, a scan for the next to last nonzero element. This could be prevented by maintaining an additional list of positions where the characteristic word is nonzero. Algorithm 4 does stay loopless, the position of the maximal element does not need to be sought. Our implementations will always use a variable holding the position of the last nonzero position in the characteristic word. This is called the “current track” in the implementations.

We will not make the equivalents of algorithm 3 loopless, as the maintenance of the additional list would render algorithm 4 slower (at least as long as mixed calls to both shall be allowed), but see section 3 for one such example.

A step in either direction has either one or three positions in the characteristic word changed. The number of transitions with one change are more frequent with higher multiplicities. The worst case occurs if all multiplicities are one (corresponding to binary words), when (about) half of the steps involve only one transition.

The C++ implementation uses the sentinel technique to reduce the number of conditional branches.

```

1 // FILE: src/comb/mixedradix-subset-lex.h
2 class mixedradix_subset_lex
3 {
4     ulong n_; // number of digits (n kinds of elements in multiset)
5     ulong tr_; // aux: current track
6     ulong *a_; // digits of mixed radix number
7               // (multiplicities in subset).
8     ulong *m1_; // nines (radix minus one) for each digit
9               // (multiplicity of kind k in superset).
10
11     mixedradix_subset_lex(ulong n, ulong mm, const ulong *m=0)
12     {
13         n_ = n;
14         a_ = new ulong[n_+2]; // two sentinels, one left, one right
15         a_[0] = 1; a_[n_+1] = 1;
16         ++a_; // nota bene
17         m1_ = new ulong[n_+2];
18         m1_[0] = 0; m1_[n_+1] = 0; // sentinel with n==0
19         ++m1_; // nota bene
20
21         mixedradix_init(n_, mm, m, m1_); // set up m1[], omitted
22
23         first();
24     }

```

The omitted routine `mixedradix_init()` sets all elements of the array `m1[]` to `mm` (fixed radix case), unless the pointer `m` is nonzero, then the elements behind `m` are copied into `m1[]` (mixed radix case).

```

1     void first()
2     {
3         for (ulong k=0; k<n_; ++k) a_[k] = 0;
4         tr_ = 0; // start by looking at leftmost (zero) digit
5     }
6
7     void last()
8     {
9         for (ulong k=0; k<n_; ++k) a_[k] = 0;

```

```

10     ulong n1 = ( n_ ? n_ - 1 : 0 );
11     a_[n1] = m1_[n1];
12     tr_ = n1;
13 }

```

Note the while-loop in the next() method.

```

1     bool next()
2     {
3         ulong j = tr_;
4         if ( a_[j] < m1_[j] ) // easy case 1: increment
5         {
6             a_[j] += 1;
7             return true;
8         }
9         // here a_[j] == m1_[j]
10        if ( j+1 < n_ ) // easy case 2: append (move track to the right)
11        {
12            ++j;
13            a_[j] = 1;
14            tr_ = j;
15            return true;
16        }
17        a_[j] = 0;
18        // find first nonzero digit to the left:
19        --j;
20        while ( a_[j] == 0 ) { --j; } // may read sentinel a_-1]
21        if ( (long)j < 0 ) return false; // current is last
22        a_[j] -= 1; // decrement digit to the left
23        ++j;
24        a_[j] = 1;
25        tr_ = j;
26        return true;
27    }
28 }

```

The method prev() is indeed loopless:

```

1     bool prev()
2     {
3         ulong j = tr_;
4         if ( a_[j] > 1 ) // easy case 1: decrement
5         {
6             a_[j] -= 1;
7             return true;
8         }
9         else
10        {
11            if ( tr_ == 0 )
12            {
13                if ( a_[0] == 0 ) return false; // current is first
14                a_[0] = 0; // now word is first (all zero)
15                return true;
16            }
17            a_[j] = 0;
18            --j; // now looking at next track to the left
19            if ( a_[j] == m1_[j] ) // easy case 2: move track to left
20            {
21                tr_ = j; // move track one left
22            }
23            else
24            {
25                a_[j] += 1; // increment digit to the left
26                j = n_ - 1;
27                a_[j] = m1_[j]; // set rightmost digit = nine
28                tr_ = j; // move to rightmost track
29            }
30            return true;
31        }
32    }
33 }

```

	colex	Gray code	subset-lex	iset
0:	[. . . .]	[4 . 2 .]	[. . . .]	{ }
1:	[1 . . .]	[3 . 2 .]	[1 . . .]	{ 0 }
2:	[2 . . .]	[2 . 2 .]	[2 . . .]	{ 0 }
3:	[3 . . .]	[1 . 2 .]	[3 . . .]	{ 0 }
4:	[4 . . .]	[. . 2 .]	[4 . . .]	{ 0 }
5:	[. 1 . .]	[. . 1 .]	[4 . 1 .]	{ 0, 2 }
6:	[. 2 . .]	[1 . 1 .]	[4 . 2 .]	{ 0, 2 }
7:	[. 3 . .]	[2 . 1 .]	[4 . . 1]	{ 0, 3 }
8:	[. . 1 .]	[3 . 1 .]	[3 . 1 .]	{ 0, 2 }
9:	[1 . 1 .]	[4 . 1 .]	[3 . 2 .]	{ 0, 2 }
10:	[2 . 1 .]	[4 . . .]	[3 . . 1]	{ 0, 3 }
11:	[3 . 1 .]	[3 . . .]	[2 . 1 .]	{ 0, 2 }
12:	[4 . 1 .]	[2 . . .]	[2 . 2 .]	{ 0, 2 }
13:	[. . 2 .]	[1 . . .]	[2 . . 1]	{ 0, 3 }
14:	[1 . 2 .]	[. . . .]	[1 . 1 .]	{ 0, 2 }
15:	[2 . 2 .]	[. 1 . .]	[1 . 2 .]	{ 0, 2 }
16:	[3 . 2 .]	[. 2 . .]	[1 . . 1]	{ 0, 3 }
17:	[4 . 2 .]	[. 3 . .]	[. 1 . .]	{ 1 }
18:	[. . . 1]	[. 3 . 1]	[. 2 . .]	{ 1 }
19:	[1 . . 1]	[. 2 . 1]	[. 3 . .]	{ 1 }
20:	[2 . . 1]	[. 1 . 1]	[. 3 . 1]	{ 1, 3 }
21:	[3 . . 1]	[. . . 1]	[. 2 . 1]	{ 1, 3 }
22:	[4 . . 1]	[1 . . 1]	[. 1 . 1]	{ 1, 3 }
23:	[. 1 . 1]	[2 . . 1]	[. . 1 .]	{ 2 }
24:	[. 2 . 1]	[3 . . 1]	[. . 2 .]	{ 2 }
25:	[. 3 . 1]	[4 . . 1]	[. . . 1]	{ 3 }

Figure 3: Mixed radix numbers of length 4 in falling factorial base that are non-adjacent forms (NAF), in co-lexicographic order, minimal-change order (Gray code), and subset-lex order (with the sets of positions of nonzero digits).

34 }

The performance of the generator is quite satisfactory. A system based on an AMD Phenom(tm) II X4 945 processor clocked at 3.0 GHz and the GCC C++ compiler version 4.9.0 [12] were used for measuring. The updates using `next()` cost about 12 cycles for base 2 (the worst case) and 9 cycles for base 16. Using `prev()` takes about 8.5 cycles with base 2 and 4.2 cycles with base 16, the latter figure corresponding to a rate of 780 million subsets per second.

These and all following such figures are average values, obtained by measuring the total time for the generation of all words of a certain size and dividing by the number of words.

3 Loopless computation of the successor for non-adjacent forms

The method of using a list of nonzero positions in the words to obtain loopless methods for both `next()` and `prev()` has been implemented for words where no two adjacent digits are nonzero (non-adjacent forms, NAF).

In the following algorithm for the computation of the successor let a_i be the digits of the length- n NAF where $0 \leq i < n$.

Algorithm 5 (Next-NAF-SL). *Compute the successor of a non-adjacent form in subset-lex order.*

1. Let j be the position of the last nonzero digit.
2. If a_j is not maximal, increment it and return.

0:	[4 . 2 .]	(0, 4) (1, 3) (2)	[4 3 2 1 0]
1:	[3 . 2 .]	(0, 3) (1, 4) (2)	[3 4 2 0 1]
2:	[2 . 2 .]	(0, 2) (1, 4) (3)	[2 4 0 3 1]
3:	[1 . 2 .]	(0, 1) (2, 4) (3)	[1 0 4 3 2]
4:	[. . 2 .]	(0) (1) (2, 4) (3)	[0 1 4 3 2]
5:	[. . 1 .]	(0) (1) (2, 3) (4)	[0 1 3 2 4]
6:	[1 . 1 .]	(0, 1) (2, 3) (4)	[1 0 3 2 4]
7:	[2 . 1 .]	(0, 2) (1, 3) (4)	[2 3 0 1 4]
8:	[3 . 1 .]	(0, 3) (1, 2) (4)	[3 2 1 0 4]
9:	[4 . 1 .]	(0, 4) (1, 2) (3)	[4 2 1 3 0]
10:	[4 . . .]	(0, 4) (1) (2) (3)	[4 1 2 3 0]
11:	[3 . . .]	(0, 3) (1) (2) (4)	[3 1 2 0 4]
12:	[2 . . .]	(0, 2) (1) (3) (4)	[2 1 0 3 4]
13:	[1 . . .]	(0, 1) (2) (3) (4)	[1 0 2 3 4]
14:	[. . . .]	(0) (1) (2) (3) (4)	[0 1 2 3 4]
15:	[. 1 . .]	(0) (1, 2) (3) (4)	[0 2 1 3 4]
16:	[. 2 . .]	(0) (1, 3) (2) (4)	[0 3 2 1 4]
17:	[. 3 . .]	(0) (1, 4) (2) (3)	[0 4 2 3 1]
18:	[. 3 . 1]	(0) (1, 4) (2, 3)	[0 4 3 2 1]
19:	[. 2 . 1]	(0) (1, 3) (2, 4)	[0 3 4 1 2]
20:	[. 1 . 1]	(0) (1, 2) (3, 4)	[0 2 1 4 3]
21:	[. . . 1]	(0) (1) (2) (3, 4)	[0 1 2 4 3]
22:	[1 . . 1]	(0, 1) (2) (3, 4)	[1 0 2 4 3]
23:	[2 . . 1]	(0, 2) (1) (3, 4)	[2 1 0 4 3]
24:	[3 . . 1]	(0, 3) (1) (2, 4)	[3 1 4 0 2]
25:	[4 . . 1]	(0, 4) (1) (2, 3)	[4 1 3 2 0]

Figure 4: Gray code for the length-4 non-adjacent forms in falling factorial base (left), together with the corresponding involutions in cycle form (middle) and in array form (right).

3. If $j + 2 < n$, set $a_{j+2} = 1$ (append new digit) and return.
4. Set $a_j = 0$ (set last nonzero digit to zero).
5. If $j + 1 < n$, set $a_{j+1} = 1$ (move digit right) and return.
6. If there is just one nonzero digit, stop.
7. Let k be the position of the nearest nonzero digit left of j .
8. Set $a_k = a_k - 1$ (decrement digit to the left).
9. If $a_k = 0$, set $a_{k+1} = 1$ (move digit right) and return.
10. Otherwise, $a_{k+2} = 1$ (append digit two positions to the right).

In the following implementation the array `iset[]` holds the positions of the nonzero digits. We only read the last or second last element, the adjustments of the length (variable `ni`) have been left out in the description above. The implementation handles all $n \geq 0$ correctly, for the all-zero word `iset[]` is of length one and its only element points to the leftmost digit.

```

1 // FILE: src/comb/mixedradix-naf-subset-lex.h
2 class mixedradix_naf_subset_lex
3 {
4     ulong *iset_; // Set of positions of nonzero digits
5     ulong *a_; // digits
6     ulong *m1_; // nines (radix minus one) for each digit
7     ulong ni_; // number of elements in iset[]
8     ulong n_; // number of digits
9
10    void first()
11    {
12        for (ulong k=0; k<n_; ++k) a_[k] = 0;
13
14        // iset[] initially with one element zero:
15        iset_[0] = 0;
16        ni_ = 1;

```

```

17
18     if ( n_==0 ) // make things work for n == 0
19     {
20         m1_[0] = 0;
21         a_[0] = 0;
22     }
23 }

```

The computation of the successor is

```

1     bool next()
2     {
3         ulong j = iset_[ni_-1];
4         const ulong aj = a_[j] + 1;
5         if ( aj <= m1_[j] ) // can increment last digit
6         {
7             a_[j] = aj;
8             return true;
9         }
10
11         if ( j + 2 < n_ ) // can append new digit
12         {
13             iset_[ni_] = j + 2;
14             a_[j+2] = 1; // assume all m1[] are nonzero
15             ++ni_;
16             return true;
17         }
18
19         a_[j] = 0; // set last nonzero digit to zero
20
21         if ( j + 1 < n_ ) // can move last digit to the right
22         {
23             a_[j+1] = 1;
24             iset_[ni_-1] = j + 1;
25             return true;
26         }
27
28         if ( ni_ == 1 ) return false; // current is last
29
30
31         // Now we look to the left:
32         const ulong k = iset_[ni_-2]; // nearest nonzero digit to the left
33         const ulong ak = a_[k] - 1; // decrement digit to the left
34         a_[k] = ak;
35         if ( ak == 0 ) // move digit one to the right
36         {
37             a_[k+1] = 1;
38             iset_[ni_-2] = k + 1;
39             --ni_;
40             return true;
41         }
42         else // append digit two positions to the right
43         {
44             a_[k+2] = 1;
45             iset_[ni_-1] = k + 2;
46             return true;
47         }
48     }

```

The update via `next()` takes about 7 cycles. The updates for co-lexicographic order and the Gray code respectively take about 15 and 21 cycles.

We note that the falling factorial NAFs of length n are one-to-one with involutions (self-inverse permutations) of $n + 1$ elements. Process the NAF from left to right; for $a_i = 0$ let the next unused element of the permutation be a fixed point (and mark it as used); for $a_i \neq 0$ put the next unused element in a cycle with the a_i th unused element (and mark both as used). A (simplistic) implementation of this method is given in the program `demo/comb/perm-involution-naf-demo.cc`.

The binary NAFs of length 8 are shown in figure 5. Algorithms for the generation of these NAFs as binary words for both lexicographic order and the Gray code are

	colex	Gray code	subset-lex
0:1..1..1	1..... = { 0 }
1:1	.1..1...	1.1..... = { 0, 2 }
2:1.	.1..1.1.	1.1.1... = { 0, 2, 4 }
3:1..	.1..1..1.	1.1.1.1. = { 0, 2, 4, 6 }
4:1.1	.1.....	1.1.1..1 = { 0, 2, 4, 7 }
5:1...	.1.....1	1.1..1.. = { 0, 2, 5 }
6:1..1	.1..1..1	1.1..1.1 = { 0, 2, 5, 7 }
7:1.1.	.1..1.1.	1.1..1.1 = { 0, 2, 6 }
8:	...1....	.1.1.1..	1.1...1 = { 0, 2, 7 }
9:	...1...1	.1.1.1.1	1..1.... = { 0, 3 }
10:	..1..1.	.1.1..1.	1..1.1.. = { 0, 3, 5 }
11:	..1.1..	.1.1....	1..1.1.1 = { 0, 3, 5, 7 }
12:	..1.1.1	.1.1.1.	1..1.1.1 = { 0, 3, 6 }
13:	..1.....	...1..1.	1..1..1 = { 0, 3, 7 }
14:	..1....1	...1....	1...1... = { 0, 4 }
15:	..1...1.	...1..1.	1...1.1. = { 0, 4, 6 }
16:	..1...1.	...1.1.1	1...1.1. = { 0, 4, 7 }
17:	..1..1.1	...1.1..	1...1.1. = { 0, 5 }
18:	..1.1...1...	1....1.1 = { 0, 5, 7 }
19:	..1.1.11.1.	1.....1 = { 0, 6 }
20:	..1.1.1.1	1.....1 = { 0, 7 }
21:	.1.....1..... = { 1 }
22:	.1.....11.	.1.1.... = { 1, 3 }
23:	.1....1.1.1.	.1.1.1.. = { 1, 3, 5 }
24:	.1...1..1...	.1.1.1.1 = { 1, 3, 5, 7 }
25:	.1...1.11..1.	.1.1..1. = { 1, 3, 6 }
26:	.1...1..	..1.1.1.	.1.1..1. = { 1, 3, 7 }
27:	.1..1.1.	..1.1...	.1..1... = { 1, 4 }
28:	.1..1.1.	..1.1.1.	.1..1.1. = { 1, 4, 6 }
29:	.1.1....	..1..1.	.1..1.1. = { 1, 4, 7 }
30:	.1.1...1	..1.....	.1..1.. = { 1, 5 }
31:	.1.1..1.	..1....1	.1..1.1. = { 1, 5, 7 }
32:	.1.1.1..	..1..1.1	.1...1. = { 1, 6 }
33:	.1.1.1.1	..1..1..	.1....1 = { 1, 7 }
34:	1.....	1.1..1..	..1..... = { 2 }
35:	1.....1	1.1..1.1	..1.1... = { 2, 4 }
36:	1....1.	1.1....1	..1.1.1. = { 2, 4, 6 }
37:	1....1..	1.1.....	..1.1.1. = { 2, 4, 7 }
38:	1...1.1	1.1...1.	..1.1.. = { 2, 5 }
39:	1...1...	1.1.1.1.	..1.1.1. = { 2, 5, 7 }
40:	1...1..1	1.1.1...	..1...1. = { 2, 6 }
41:	1...1.1.	1.1.1..1	..1...1. = { 2, 7 }
42:	1..1....	1..1..1.	...1.... = { 3 }
43:	1..1...1	1..1...	...1.1.. = { 3, 5 }
44:	1..1..1.	1..1.1.	...1.1.1 = { 3, 5, 7 }
45:	1..1.1..	1.....1.	...1..1. = { 3, 6 }
46:	1..1.1.1	1.....	...1..1. = { 3, 7 }
47:	1.1.....	1.....11... = { 4 }
48:	1.1....1	1....1.11.1. = { 4, 6 }
49:	1.1...1.	1....1..1.1. = { 4, 7 }
50:	1.1..1..	1..1.1..1.. = { 5 }
51:	1.1..1.1	1..1.1.11.1. = { 5, 7 }
52:	1.1.1...	1..1..1.1. = { 6 }
53:	1.1.1..1	1..1....1 = { 7 }
54:	1.1.1.1.	1..1..1.	

Figure 5: Binary non-adjacent forms of length 8 in co-lexicographic order, minimal-change order (Gray code), and subset-lex order (with the corresponding sets without consecutive elements).

given in [2, p.75-77]. Here we give the implementations for computing successor and predecessor for the nonempty NAFs in subset-lex order.

The routine for the successor needs to start with the word that has a single set bit at the highest position of the desired word length.

```

1 // FILE: src/bits/fibrep-subset-lexrev.h
2 ulong next_subset_lexrev_fib(ulong x)
3 {
4     ulong x0 = x & -x; // lowest bit
5     ulong xs = x0 >> 2;
6     if ( xs != 0 ) // easy case: set bit right of lowest bit
7     {
8         x |= xs;
9         return x;
10    }
11    else // lowest bit at index 0 or 1
12    {
13        if ( x0 == 2 ) // at index 1
14        {
15            x -= 1;
16            return x;
17        }
18
19        x ^= x0; // clear lowest bit
20        x0 = x & -x; // new lowest bit ...
21        x0 >>= 1; x -= x0; // ... is moved one to the right
22        return x;
23    }
24 }

```

The all-zero word is returned as successor of the word whose value is 1. The routine for the predecessor can be started with the all-zero word.

```

1 ulong prev_subset_lexrev_fib(ulong x)
2 {
3     ulong x0 = x & -x; // lowest bit
4     if ( x & (x0<<2) ) // easy case: next higher bit is set
5     {
6         x ^= x0; // clear lowest bit
7         return x;
8     }
9     else
10    {
11        x += x0; // move lowest bit to the left and
12        x |= (x0!=1); // set rightmost bit unless blocked by next bit
13        return x;
14    }
15 }

```

0:	[1 1 1 1 1 1 1]
1:1	[1 1 1 1 1 2]
2:1.	[1 1 1 1 2 1]
3:11	[1 1 1 1 3]
4:	...1..	[1 1 1 2 1 1]
5:	...1.1	[1 1 1 2 2]
6:	...11.	[1 1 1 3 1]
7:	...111	[1 1 1 4]
8:	..1...	[1 1 2 1 1 1]
9:	..1...1	[1 1 2 1 2]
10:	..1.1.	[1 1 2 2 1]
11:	..1.11	[1 1 2 3]
12:	..11..	[1 1 3 1 1]
13:	..11.1	[1 1 3 2]
14:	..111.	[1 1 4 1]
15:	..1111	[1 1 5]
16:	.1....	[1 2 1 1 1 1]
17:	.1...1	[1 2 1 1 2]
18:	.1..1.	[1 2 1 2 1]
19:	.1..11	[1 2 1 3]
20:	.1.1..	[1 2 2 1 1]
21:	.1.1.1	[1 2 2 2]
22:	.1.11.	[1 2 3 1]
23:	.1.111	[1 2 4]
24:	.11...	[1 3 1 1 1]
25:	.11...1	[1 3 1 2]
26:	.11.1.	[1 3 2 1]
27:	.11.11	[1 3 3]
28:	.111..	[1 4 1 1]
29:	.111.1	[1 4 2]
30:	.1111.	[1 5 1]
31:	.11111	[1 6]
32:	1.....	[2 1 1 1 1 1]
33:	1....1	[2 1 1 1 2]
34:	1...1.	[2 1 1 2 1]
35:	1...11	[2 1 1 3]
36:	1..1..	[2 1 2 1 1]
37:	1..1.1	[2 1 2 2]
38:	1..11.	[2 1 3 1]
39:	1..111	[2 1 4]
40:	1.1...	[2 2 1 1 1]
41:	1.1...1	[2 2 1 2]
42:	1.1.1.	[2 2 2 1]
43:	1.1.11	[2 2 3]
44:	1.11..	[2 3 1 1]
45:	1.11.1	[2 3 2]
46:	1.111.	[2 4 1]
47:	1.1111	[2 5]
48:	11....	[3 1 1 1 1]
49:	11...1	[3 1 1 2]
50:	11..1.	[3 1 2 1]
51:	11..11	[3 1 3]
52:	11.1..	[3 2 1 1]
53:	11.1.1	[3 2 2]
54:	11.11.	[3 3 1]
55:	11.111	[3 4]
56:	111...	[4 1 1 1]
57:	111...1	[4 1 2]
58:	111.1.	[4 2 1]
59:	111.11	[4 3]
60:	1111..	[5 1 1]
61:	1111.1	[5 2]
62:	11111.	[6 1]
63:	111111	[7]

Figure 6: The compositions of 7 together with their run-length encodings as binary words (where dots denote zeros), lexicographic order. A succession of k ones, followed by a zero, in the run-length encoding stands for a part $k+1$, one trailing zero is implied.

4 Compositions

One of the most simple algorithms in combinatorial generation may be the computation of the successor of a composition represented as a list of parts for the lexicographic ordering. In the following let z be the last element of the composition.

Algorithm 6 (Next-Comp). *Compute the successor of a composition in lexicographic order.*

1. If there is just one part, stop (this is the last composition).
2. Add 1 to the second last part (and remove z) and append $z - 1$ ones at the end.

Algorithm 7 (Prev-Comp). *Compute the predecessor of a composition in lexicographic order.*

1. If the number of parts is maximal (composition into all ones), stop.
2. If $z > 1$, replace z by $z - 1, 1$ (move one unit right); return
3. Otherwise, replace the tail $y, 1^q$ (y followed by q ones) by $y - 1, q + 1$.

Figure 6 shows the compositions of 7 in lexicographic order. The corresponding run-length encodings appear in lexicographic order as well.

The algorithm for the successor can be made loopless when care is taken that only ones are left beyond the end of the current composition.

```

1 // FILE: src/comb/composition-nz.h
2 class composition_nz
3 // Compositions of n into positive parts, lexicographic order.
4 {
5     ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
6     ulong n_; // composition of n
7     ulong m_; // current composition is into m parts
8
9     ulong next()
10 // Return number of parts of generated composition.
11 // Return zero if the current is the last composition.
12 {
13     if ( m_ <= 1 ) return 0; // current is last
14
15     // [* , Y, Z] --> [* , Y+1, 1, 1, 1, ..., 1] (Z-1 trailing ones)
16     a_[m_-1] += 1;
17     const ulong z = a_[m_];
18     a_[m_] = 1;
19     // all parts a[m+1], a[m+2], ..., a[n] are already ==1
20     m_ += z - 2;
21
22     return m_;
23 }
```

Figure 7 shows two orderings for the compositions. The ordering corresponding to the (complemented) binary Gray code is shown in the left columns. We will call this order *RL-order*, as the compositions correspond to the run-lengths of the binary words in lexicographic order.

0:	111111	[7]	[7]
1:	11111.	[6 1]	1.....	[1 6]
2:	1111..	[5 1 1]	11.....	[1 1 5]
3:	1111.1	[5 2]	111....	[1 1 1 4]
4:	111..1	[4 1 2]	1111...	[1 1 1 1 3]
5:	111...	[4 1 1 1]	11111..	[1 1 1 1 1 2]
6:	111.1.	[4 2 1]	111111.	[1 1 1 1 1 1 1]
7:	111.11	[4 3]	11111.1	[1 1 1 1 2 1]
8:	11..11	[3 1 3]	1111.1.	[1 1 1 2 2]
9:	11..1.	[3 1 2 1]	1111.11	[1 1 1 2 1 1]
10:	11....	[3 1 1 1 1]	1111..1	[1 1 1 3 1]
11:	11...1	[3 1 1 2]	111.1..	[1 1 2 3]
12:	11.1.1	[3 2 2]	111.11.	[1 1 2 1 2]
13:	11.1..	[3 2 1 1]	111.111	[1 1 2 1 1 1]
14:	11.11.	[3 3 1]	111.1.1	[1 1 2 2 1]
15:	11.111	[3 4]	11..1.	[1 1 3 2]
16:	1..111	[2 1 4]	11..11	[1 1 3 1 1]
17:	1..11.	[2 1 3 1]	11...1	[1 1 4 1]
18:	1..1..	[2 1 2 1 1]	1.1...	[1 2 4]
19:	1..1.1	[2 1 2 2]	1.11..	[1 2 1 3]
20:	1....1	[2 1 1 1 2]	1.111.	[1 2 1 1 2]
21:	1.....	[2 1 1 1 1 1]	1.1111	[1 2 1 1 1 1]
22:	1...1.	[2 1 1 2 1]	1.11.1	[1 2 1 2 1]
23:	1...11	[2 1 1 3]	1.1.1.	[1 2 2 2]
24:	1.1.11	[2 2 3]	1.1.11	[1 2 2 1 1]
25:	1.1.1.	[2 2 2 1]	1.1..1	[1 2 3 1]
26:	1.1...	[2 2 1 1 1]	1..1..	[1 3 3]
27:	1.1..1	[2 2 1 2]	1..11.	[1 3 1 2]
28:	1.11.1	[2 3 2]	1..111	[1 3 1 1 1]
29:	1.11..	[2 3 1 1]	1..1.1	[1 3 2 1]
30:	1.111.	[2 4 1]	1...1.	[1 4 2]
31:	1.1111	[2 5]	1...11	[1 4 1 1]
32:	..1111	[1 1 5]	1....1	[1 5 1]
33:	..111.	[1 1 4 1]	.1....	[2 5]
34:	..11..	[1 1 3 1 1]	.11...	[2 1 4]
35:	..11.1	[1 1 3 2]	.111..	[2 1 1 3]
36:	..1..1	[1 1 2 1 2]	.1111.	[2 1 1 1 2]
37:	..1...	[1 1 2 1 1 1]	.11111	[2 1 1 1 1 1]
38:	..1.1.	[1 1 2 2 1]	.111.1	[2 1 1 2 1]
39:	..1.11	[1 1 2 3]	.11.1.	[2 1 2 2]
40:	...11	[1 1 1 1 3]	.11.11	[2 1 2 1 1]
41:	...1.	[1 1 1 1 2 1]	.11..1	[2 1 3 1]
42:	[1 1 1 1 1 1 1]	.1.1..	[2 2 3]
43:1	[1 1 1 1 1 2]	.1.11.	[2 2 1 2]
44:	...1.1	[1 1 1 2 2]	.1.111	[2 2 1 1 1]
45:	...1..	[1 1 1 2 1 1]	.1.1.1	[2 2 2 1]
46:	...11.	[1 1 1 3 1]	.1..1.	[2 3 2]
47:	...111	[1 1 1 4]	.1..11	[2 3 1 1]
48:	.1.111	[1 2 4]	.1...1	[2 4 1]
49:	.1.11.	[1 2 3 1]	..1...	[3 4]
50:	.1.1..	[1 2 2 1 1]	..11..	[3 1 3]
51:	.1.1.1	[1 2 2 2]	..111.	[3 1 1 2]
52:	.1...1	[1 2 1 1 2]	..1111	[3 1 1 1 1]
53:	.1....	[1 2 1 1 1 1]	..11.1	[3 1 2 1]
54:	.1..1.	[1 2 1 2 1]	..1.1.	[3 2 2]
55:	.1..11	[1 2 1 3]	..1.11	[3 2 1 1]
56:	..11.11	[1 3 3]	..1..1	[3 3 1]
57:	..11.1.	[1 3 2 1]	...1..	[4 3]
58:	..11...	[1 3 1 1 1]	...11.	[4 1 2]
59:	..11..1	[1 3 1 2]	...111	[4 1 1 1]
60:	..111.1	[1 4 2]	...1.1	[4 2 1]
61:	..111..	[1 4 1 1]1.	[5 2]
62:	..1111.	[1 5 1]11	[5 1 1]
63:	..11111	[1 6]1	[6 1]

Figure 7: The compositions of 7 together with their run-length encodings as binary words (where dots denote zeros), in an order corresponding to the complemented binary Gray code (left) and in subset-lex order (right). A succession of k ones, followed by a zero, in the run-length encoding stands for a part $k+1$, one trailing zero is implied (left). The roles of ones and zeros are reversed for the subset-lex order (right).

For comparison with the new algorithm we give the (loopless) algorithms. In the following let m be the number of parts in the composition and x, y, z the last three parts.

Algorithm 8 (Next-Comp-RL). *Compute the successor of a composition in RL-order.*

1. If m is odd: if $z \geq 2$, replace z by $z - 1, 1$ and return; otherwise ($z = 1$) replace $y, 1$ by $y + 1$ and return.
2. If m is even: if $y \geq 2$, replace y, z by $y - 1, 1, z$ and return; otherwise ($y = 1$) replace $x, 1, z$ by $x + 1, z$ and return.

The next algorithm is obtained from the previous simply by swapping “even” and “odd” in the description.

Algorithm 9 (Prev-Comp-RL). *Compute the predecessor of a composition in RL-order.*

1. If m is even: if $z \geq 2$, replace z by $z - 1, 1$ and return; otherwise ($z = 1$) replace $y, 1$ by $y + 1$ and return.
2. If m is odd: if $y \geq 2$, replace y, z by $y - 1, 1, z$ and return; otherwise ($y = 1$) replace $x, 1, z$ by $x + 1, z$ and return.

With each transition, at most three parts (at the end of the composition) are changed. The number of parts changes by 1 with each step².

An alternative algorithm for this ordering is given in [19, ex.12, sect.7.2.1.1, p.308], see also [24].

Now we give the algorithms for subset-lex order (the first is essentially given in [23]).

Algorithm 10 (Next-Comp-SL). *Compute the successor of a composition in subset-lex order.*

1. If $z = 1$ and there are at most two parts, stop.
2. If $z \geq 2$, replace z by $1, z - 1$ (move all but one unit to the right); return.
3. Otherwise ($z = 1$) add 1 to the third last part and remove z (move one unit two places left: $x, y, 1$ is replaced by $x + 1, y$).

For the next algorithm let y, z be the two last elements.

Algorithm 11 (Prev-Comp-SL). *Compute the predecessor of a composition in subset-lex order.*

1. If there is just one part, stop.
2. If $y = 1$, replace y, z by $z + 1$ (add z to the left); return.
3. Otherwise ($y \geq 2$) replace y, z by $y - 1, z, 1$ (move one unit two places right).

At most two parts are changed by either method and these span at most those three parts at the end of the composition. Again, the number of parts changes by 1 with each step.

Note that also the initializations are loopless for the preceding two orderings.

We give the crucial parts of the implementation.

```

1 // FILE: src/comb/composition-nz-subset-lex.h
2 class composition_nz_subset_lex
3 {
```

²See `src/comb/composition-nz-rl.h` for an implementation.


```

4     ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // composition of n
6     ulong m_; // current composition is into m parts
7
8     void first()
9     {
10        a_[0] = 0;
11        a_[1] = n_;
12        m_ = ( n_ ? 1 : 0 );
13    }
14
15    void last()
16    {
17        if ( n_ >= 2 )
18        {
19            a_[1] = n_ - 1;
20            a_[2] = 1;
21            for (ulong j=2; j<=n_; ++j) a_[j] = 1;
22            m_ = 2;
23        }
24        else
25        {
26            a_[1] = n_;
27            m_ = n_;
28        }
29    }

```

For the methods `next()` and `prev()` we give one of the two implementations found in the file. Both methods return the number of parts in the generated composition and return zero if there are no more compositions.

```

1     ulong next()
2     {
3         const ulong z = a_[m_];
4         if ( z<=1 ) // move one unit two places left
5         { //   [* , X, Y, 1] --> [* , X+1, Y]
6             if ( m_ <= 2 ) return 0; // current is last
7             m_ -= 1;
8             a_[m_-1] += 1;
9             return m_;
10        }
11        else // move all but one unit right
12        { //   [* , Y, Z] --> [* , Y, 1, Z-1]
13            a_[m_] = 1;
14            m_ += 1;
15            a_[m_] = z - 1;
16            return m_;
17        }
18    }
19
20    ulong prev()
21    {
22        if ( m_ <= 1 ) return 0; // current is first
23
24        const ulong y = a_[m_-1];
25        if ( y==1 ) // add Z to left place
26        { //   [* , 1, Z] --> [* , Z+1]
27            const ulong z = a_[m_];
28            a_[m_-1] = z + 1;
29            a_[m_] = 1;
30            m_ -= 1;
31            return m_;
32        }
33        else // move one unit two places right
34        { //   [* , Y, Z] --> [* , Y-1, Z, 1]
35            a_[m_-1] = y - 1;
36            m_ += 1;
37            // a[m] == 1 already
38            return m_;
39        }
40    }

```

The method `next()` takes about 5 cycles for lexicographic order, 9 cycles for RL-order, and 6 cycles for subset-lex order. The method `prev()` takes about 10.5 cycles

for lexicographic order and is identical in performance to `next()` for the other orders.

Note that the last parts in the successive compositions in lexicographic order give the (one-based) ruler function (sequence A001511 in [28]), see `src/comb/ruler-func1.h` for the trivial implementation, compare to `src/comb/ruler-func.h` (giving sequence A007814 in [28]) which uses the techniques from [10] and [3] as described in [19, Algorithm L, p.290].

An loopless implementation for the generation of the compositions into odd parts in subset-lex order is given in `src/comb/composition-nz-odd-subset-lex.h`.

Ranking and unranking

An unranking algorithm for the subset-lex order is obtained by observing (see figure 7) that the composition into one part has rank 0 and otherwise the first part and the remaining parts are easily determined.

Algorithm 12 (Unrank-Comp-SL). *Recursive routine $U(r, n, C[], m)$ for the computation of the composition $C[]$ of n with rank r in subset-lex order. The auxiliary variable $m \geq 0$ is the index of the part about to be written in $C[]$. The initial call is $F(r, n, C[], 0)$.*

1. If $r = 0$, set $C[m] = n$ and return $m + 1$ (the number of parts).
2. Set $f = 0$ (the tentative first part).
3. Set $f = f + 1$ and $t = 2^{n-f-1}$.
4. If $r < t$ (can use part f), set $C[m] = f$ and return $F(r, n - f, C[], m + 1)$.
5. Set $r = r - t$ and go to step 3.

We give an iterative implementation.

```

1 // FILE: src/comb/composition-nz-rank.cc
2 ulong composition_nz_subset_lex_unrank(ulong r, ulong *x, ulong n)
3 {
4     if ( n==0 ) return 0;
5
6     ulong m = 0;
7     while ( true )
8     {
9         if ( r==0 ) // composition into one part
10        {
11            x[m++] = n;
12            return m;
13        }
14        r -= 1;
15        ulong t = 1UL << (n-1);
16        for (ulong f=1; f<n; ++f) // find first part f >= 1
17        {
18            t >>= 1; // == 2**(n-f-1)
19            // == number of compositions of n with first part f
20
21            if ( r < t ) // first part is f
22            {
23                x[m++] = f;
24                n -= f;
25                break;
26            }
27            r -= t;
28        }
29    }
30 }
```

The following algorithm for computing the rank is modeled as inverse of the method above.

Algorithm 13 (Rank-Comp-SL). *Computation of the rank r of a composition $C[]$ of n in subset-lex order.*

1. Set $r = 0$ and $e = 0$ (position of part under consideration).
2. Set $f = C[e]$ (part under consideration).
3. If $f = n$, return r .
4. Set $r = r + 1$, $t = 2^{n-1}$, and $n = n - f$.
5. While $f > 1$, set $t = t/2$, $r = r + t$, and $f = f - 1$.
6. Set $e = e + 1$ and go to step 2

An implementation is

```

1  ulong composition_nz_subset_lex_rank(const ulong *x, ulong m, ulong n)
2  {
3      ulong r = 0; // rank
4      ulong e = 0; // position of first part
5      while ( e < m )
6      {
7          ulong f = x[e];
8          if ( f==n ) return r;
9          r += 1;
10         ulong t = 1UL << (n-1);
11         n -= f;
12         while ( f > 1 )
13         {
14             t >>= 1;
15             r += t;
16             f -= 1;
17         }
18         e += 1;
19     }
20     return r; // return r==0 for the empty composition
21 }
```

Conversion functions between binary words in lexicographic order and subset-lex order are shown in [2, pp.71-72].

```

1  // FILE: src/bits/bitlex.h
2  ulong negidx2lexrev(ulong k)
3  {
4      ulong z = 0;
5      ulong h = highest_one(k);
6      while ( k )
7      {
8          while ( 0==(h&k) ) h >>= 1;
9          z ^= h;
10         ++k;
11         k &= h - 1;
12     }
13     return z;
14 }
15 }
16
17 ulong lexrev2negidx(ulong x)
18 {
19     if ( 0==x ) return 0;
20     ulong h = x & -x; // lowest one
21     ulong r = (h-1);
22     while ( x^=h )
23     {
24         r += (h-1);
25         h = x & -x; // next higher one
26     }
27     r += h; // highest bit
28     return r;
29 }
```

Based on these, alternative ranking and unranking functions can be given³.

Ranking and unranking methods for the compositions into odd parts can be obtained by replacing 2^{n-f-1} (number of compositions of n with first part f) in the algorithms 12 and 13 by the Fibonacci numbers F_{n-f-1} (number of compositions of n into odd parts with first part f), where $n \geq 1$, $f < n$, and f odd (see sequence A242086 in [28]).

5 Partitions as weakly increasing lists of parts

We now give algorithms for the computation of the successor for partitions represented as weakly increasing lists of parts. The generation of all partitions in this representation is the subject of [16].

5.1 All partitions

Figure 8 shows the partitions of 11 as weakly increasing lists of parts, in lexicographic order (left) and in subset-lex order (right). We first give a description of the computation of the successor in lexicographic order. The three last parts of the partition are denoted by x , y , and z .

Algorithm 14 (Next-Part-Asc). *Compute the successor of a partition in lexicographic order.*

1. If there is just one part, stop.
2. If $z - 1 < y + 1$, replace y, z by $y + z$; return.
3. Otherwise, change y to $y + 1$ and append parts $y + 1$ as long as there are at least $y + 1$ units left.
4. Add the remaining units to the last part.

The second step of the algorithm can be merged into the other steps, as it is a special case of them.

```

1 // FILE: src/comb/partition-asc.h
2 class partition_asc
3 {
4     ulong *a_; // partition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // integer partitions of n
6     ulong m_; // current partition has m parts

```

The implementation is correct for all $n \geq 0$, for $n = 0$ the empty list of parts is generated.

```

1     ulong next()
2     // Return number of parts of generated partition.
3     // Return zero if the current partition is the last.
4     {
5         if ( m_ <= 1 ) return 0; // current is last
6
7         ulong z1 = a_[m_] - 1; // take one unit from last part
8         m_ -= 1;
9         const ulong y1 = a_[m_] + 1; // add one unit to previous part
10
11         while ( y1 <= z1 ) // can put part Y+1
12         {
13             a_[m_] = y1;

```

³See `src/comb/composition-nz-rank.cc` where the corresponding routines for all three orders shown here are implemented.

1:	[1 1 1 1 1 1 1 1 1 1 1 1]	[11]
2:	[1 1 1 1 1 1 1 1 1 1 2]	[1 10]
3:	[1 1 1 1 1 1 1 1 1 1 3]	[1 1 9]
4:	[1 1 1 1 1 1 1 1 1 2 2]	[1 1 1 8]
5:	[1 1 1 1 1 1 1 1 1 4]	[1 1 1 1 7]
6:	[1 1 1 1 1 1 1 1 2 3]	[1 1 1 1 1 6]
7:	[1 1 1 1 1 1 1 1 5]	[1 1 1 1 1 1 5]
8:	[1 1 1 1 1 1 2 2 2]	[1 1 1 1 1 1 1 4]
9:	[1 1 1 1 1 1 2 4]	[1 1 1 1 1 1 1 1 3]
10:	[1 1 1 1 1 1 3 3]	[1 1 1 1 1 1 1 1 1 2]
11:	[1 1 1 1 1 1 6]	[1 1 1 1 1 1 1 1 1 1 1]
12:	[1 1 1 1 1 2 2 3]	[1 1 1 1 1 1 1 1 2 2]
13:	[1 1 1 1 1 2 5]	[1 1 1 1 1 1 1 2 3]
14:	[1 1 1 1 1 3 4]	[1 1 1 1 1 1 2 4]
15:	[1 1 1 1 1 7]	[1 1 1 1 1 1 2 2 2]
16:	[1 1 1 1 2 2 2 2]	[1 1 1 1 1 1 3 3]
17:	[1 1 1 1 2 2 4]	[1 1 1 1 1 2 5]
18:	[1 1 1 1 2 3 3]	[1 1 1 1 1 2 2 3]
19:	[1 1 1 1 2 6]	[1 1 1 1 1 3 4]
20:	[1 1 1 1 3 5]	[1 1 1 1 2 6]
21:	[1 1 1 1 4 4]	[1 1 1 1 2 2 4]
22:	[1 1 1 1 8]	[1 1 1 1 2 2 2 2]
23:	[1 1 1 2 2 2 3]	[1 1 1 1 2 3 3]
24:	[1 1 1 2 2 5]	[1 1 1 1 3 5]
25:	[1 1 1 2 3 4]	[1 1 1 1 4 4]
26:	[1 1 1 2 7]	[1 1 1 2 7]
27:	[1 1 1 3 3 3]	[1 1 1 2 2 5]
28:	[1 1 1 3 6]	[1 1 1 2 2 2 3]
29:	[1 1 1 4 5]	[1 1 1 2 3 4]
30:	[1 1 1 9]	[1 1 1 3 6]
31:	[1 1 2 2 2 2 2]	[1 1 1 3 3 3]
32:	[1 1 2 2 2 4]	[1 1 1 4 5]
33:	[1 1 2 2 3 3]	[1 2 8]
34:	[1 1 2 2 6]	[1 2 2 6]
35:	[1 1 2 3 5]	[1 2 2 2 4]
36:	[1 1 2 4 4]	[1 2 2 2 2 2]
37:	[1 1 2 8]	[1 2 2 3 3]
38:	[1 1 3 3 4]	[1 2 3 5]
39:	[1 1 3 7]	[1 2 4 4]
40:	[1 1 4 6]	[1 3 7]
41:	[1 1 5 5]	[1 3 3 4]
42:	[1 1 10]	[1 4 6]
43:	[1 2 2 2 2 3]	[1 5 5]
44:	[1 2 2 2 5]	[2 9]
45:	[1 2 2 3 4]	[2 2 7]
46:	[1 2 2 7]	[2 2 2 5]
47:	[1 2 3 3 3]	[2 2 2 2 3]
48:	[1 2 3 6]	[2 2 3 4]
49:	[1 2 4 5]	[2 3 6]
50:	[1 2 9]	[2 3 3 3]
51:	[1 3 3 5]	[2 4 5]
52:	[1 3 4 4]	[3 8]
53:	[1 3 8]	[3 3 5]
54:	[1 4 7]	[3 4 4]
55:	[1 5 6]	[4 7]
56:	[1 11]	[5 6]

Figure 8: The partitions of 11 as weakly increasing lists of parts, lexicographic order (left) and subset-lex order (right).

```

14         z1 -= y1;
15         m_ += 1;
16     }
17     a_[m_] = y1 + z1; // add remaining units to last part
18
19     return m_;
20 }

```

The computation of the successor in subset-lex order is loopless.

Algorithm 15 (Next-Part-Asc-SL). *Compute the successor of a partition in subset-lex order. A sentinel zero shall precede list of elements.*

1. If $z \geq 2y$, replace y, z by $y, y, z - y$ (extend to the right); return.
2. If $z - 1 \geq y + 1$, replace y, z by $y + 1, z - 1$ (add one unit to the left); return.
3. If $z = 1$ (the all-ones partition) do the following. Stop if the number of parts is ≤ 3 , otherwise replace the tail $[1, 1, 1, 1]$ by $[2, 2]$ and return.
4. If the number of parts is 2, stop.
5. Replace x, y, z by $x + 1, y + z - 1$ (add one unit to second left, add rest to end) and return.

Note that with each update all but the last two parts are the same as in the predecessor. This can be an advantage over the lexicographic order for computations where partial results for prefixes can be reused.

```

1 // FILE: src/comb/partition-asc-subset-lex.h
2 class partition_asc_subset_lex
3 {
4     ulong *a_; // partition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // partition of n
6     ulong m_; // current partition has m parts
7
8     explicit partition_asc_subset_lex(ulong n)
9     {
10         n_ = n;
11         a_ = new ulong[n_+1+(n_==0)];
12         // sentinel a[0] set in first()
13         first();
14     }
15
16     void first()
17     {
18         a_[0] = 1; // sentinel: read (once) by test for right-extension
19         a_[1] = n_ + (n_==0); // use partitions of n=1 for n=0 internally
20         m_ = 1;
21     }

```

The implementation is again correct for all $n \geq 0$.

```

1     ulong next()
2     // Loopless algorithm.
3     {
4         ulong y = a_[m_-1]; // may read sentinel a[0]
5         ulong z = a_[m_];
6
7         if ( z >= 2*y ) // extend to the right:
8         { // [* , Y, Z] --> [* , Y, Y, Z-Y]
9             a_[m_] = y;
10            a_[m_+1] = z - y; // >= y
11            ++m_;
12            return m_;
13        }
14
15        z -= 1; y += 1;
16        if ( z >= y ) // add one unit to the left:
17        { // [* , Y, Z] --> [* , Y+1, Z-1]
18            a_[m_-1] = y;
19            a_[m_] = z;

```

```

20     return m_;
21 }
22
23 if ( z == 0 ) // all-ones partition
24 {
25     if ( n_ <= 3 ) return 0; // current is last
26
27     // [1, ..., 1, 1, 1, 1] --> [1, ..., 2, 2]
28     m_ -= 2;
29     a_[m_] = 2; a_[m_-1] = 2;
30     return m_;
31 }
32
33 // add one unit to second left, add rest to end:
34 // [* , X, Y, Z] --> [* , X+1, Y+Z-1]
35 a_[m_-2] += 1;
36 a_[m_-1] += z;
37 m_ -= 1;
38 m_ -= (m_ == 1); // last if partition is into one part
39 return m_;
40 }

```

The updates via `next()` take about 15 cycles for lexicographic order and about 13 cycles for subset-lex order.

5.2 Partitions into odd and into distinct parts

It is not very difficult to adapt the algorithms to specialized partitions. We give the algorithms for partitions into distinct and odd parts by their implementations.

Computation of the successor for partitions into distinct parts in lexicographic order:

```

1 // FILE: src/comb/partition-dist-asc.h
2     ulong next()
3     {
4         if ( m_ <= 1 ) return 0; // current is last
5
6         ulong s = a_[m_] + a_[m_-1];
7         ulong k = a_[m_-1] + 1;
8         m_ -= 1;
9         // split s into k, k+1, k+2, ..., y, z where z >= y + 1:
10        while ( s >= ( k + (k+1) ) )
11        {
12            a_[m_] = k;
13            s -= k;
14            k += 1;
15            m_ += 1;
16        }
17
18        a_[m_] = s;
19        return m_;
20    }
21 }

```

Computation of the successor for partitions into odd parts in lexicographic order:

```

1 // FILE: src/comb/partition-odd-asc.h
2     ulong next()
3     {
4         const ulong z = a_[m_]; // can read sentinel a[0] if n==0
5         const ulong y = a_[m_-1]; // can read sentinel a[0] (a[-1] for n==0)
6         ulong s = y + z; // sum of parts we scan over
7
8         ulong k; // min value of next term
9         if ( z >= y+4 ) // add last 2 terms
10        {
11            if ( m_ == 1 ) return 0; // current is last
12            k = y + 2;
13            a_[m_-1] = k;
14            s -= k;
15        }

```

0:	[19]	[19]
1:	[1 18]	[1 1 17]
2:	[1 2 16]	[1 1 1 1 15]
3:	[1 2 3 13]	[1 1 1 1 1 1 13]
4:	[1 2 3 4 9]	[1 1 1 1 1 1 1 1 11]
5:	[1 2 3 5 8]	[1 1 1 1 1 1 1 1 1 1 9]
6:	[1 2 3 6 7]	[1 1 1 1 1 1 1 1 1 1 7]
7:	[1 2 4 12]	[1 1 1 1 1 1 1 1 1 1 1 5]
8:	[1 2 4 5 7]	[1 1 1 1 1 1 1 1 1 1 1 1 3]
9:	[1 2 5 11]	[1 1 1 1 1 1 1 1 1 1 1 1 1 1]
10:	[1 2 6 10]	[1 1 1 1 1 1 1 1 1 1 1 1 1 3 3]
11:	[1 2 7 9]	[1 1 1 1 1 1 1 1 1 1 1 1 3 5]
12:	[1 3 15]	[1 1 1 1 1 1 1 1 1 1 3 3 3]
13:	[1 3 4 11]	[1 1 1 1 1 1 1 1 1 1 3 7]
14:	[1 3 4 5 6]	[1 1 1 1 1 1 1 1 1 1 5 5]
15:	[1 3 5 10]	[1 1 1 1 1 1 1 1 1 3 3 5]
16:	[1 3 6 9]	[1 1 1 1 1 1 1 1 3 9]
17:	[1 3 7 8]	[1 1 1 1 1 1 1 1 3 3 3 3]
18:	[1 4 14]	[1 1 1 1 1 1 1 5 7]
19:	[1 4 5 9]	[1 1 1 1 1 1 3 3 7]
20:	[1 4 6 8]	[1 1 1 1 1 1 3 5 5]
21:	[1 5 13]	[1 1 1 1 1 3 11]
22:	[1 5 6 7]	[1 1 1 1 1 3 3 3 5]
23:	[1 6 12]	[1 1 1 1 1 5 9]
24:	[1 7 11]	[1 1 1 1 1 7 7]
25:	[1 8 10]	[1 1 1 1 3 3 9]
26:	[2 17]	[1 1 1 1 3 3 3 3 3]
27:	[2 3 14]	[1 1 1 1 3 5 7]
28:	[2 3 4 10]	[1 1 1 1 5 5 5]
29:	[2 3 5 9]	[1 1 1 3 13]
30:	[2 3 6 8]	[1 1 1 3 3 3 7]
31:	[2 4 13]	[1 1 1 3 3 5 5]
32:	[2 4 5 8]	[1 1 1 5 11]
33:	[2 4 6 7]	[1 1 1 7 9]
34:	[2 5 12]	[1 1 3 3 11]
35:	[2 6 11]	[1 1 3 3 3 3 5]
36:	[2 7 10]	[1 1 3 5 9]
37:	[2 8 9]	[1 1 3 7 7]
38:	[3 16]	[1 1 5 5 7]
39:	[3 4 12]	[1 3 15]
40:	[3 4 5 7]	[1 3 3 3 9]
41:	[3 5 11]	[1 3 3 3 3 3 3]
42:	[3 6 10]	[1 3 3 5 7]
43:	[3 7 9]	[1 3 5 5 5]
44:	[4 15]	[1 5 13]
45:	[4 5 10]	[1 7 11]
46:	[4 6 9]	[1 9 9]
47:	[4 7 8]	[3 3 13]
48:	[5 14]	[3 3 3 3 7]
49:	[5 6 8]	[3 3 3 5 5]
50:	[6 13]	[3 5 11]
51:	[7 12]	[3 7 9]
52:	[8 11]	[5 5 9]
53:	[9 10]	[5 7 7]

Figure 9: The partitions of 19 into distinct parts (left) and odd parts (right) in subset-lex order.


```

16     else // add last 3 terms
17     {
18         if ( m_ <= 2 ) return 0; // current is last
19         const ulong x = a_[m_-2];
20         s += x;
21         k = x + 2;
22         m_ -= 2;
23     }
24
25     const ulong k2 = k + k;
26     while ( s >= k2 + k )
27     {
28         a_[m_] = k; s -= k; m_ += 1;
29         a_[m_] = k; s -= k; m_ += 1;
30     }
31
32     a_[m_] = s;
33     return m_;
34 }

```

All routines in this section return the number of parts in the generated partition and return zero if there are no more partitions.

Computation of the successor for partitions into distinct parts in subset-lex order:

```

1 // FILE: src/comb/partition-dist-asc-subset-lex.h
2 ulong next()
3 // Loopless algorithm.
4 {
5     ulong y = a_[m_-1]; // may read sentinel a[0]
6     ulong z = a_[m_];
7
8     if ( z >= 2*y + 3 ) // can extend to the right
9     { // [* , Y, Z] --> [* , Y, Y+1, Z-1]
10        y += 1;
11        a_[m_] = y;
12        a_[m_+1] = z - y; // >= y
13        ++m_;
14        return m_;
15    }
16    else // add to the left
17    {
18        z -= 1; y += 1;
19
20        if ( z > y ) // add one unit to the left
21        { // [* , Y, Z] --> [* , Y+1, Z-1]
22
23            if ( m_<=1 ) return 0; // current is last
24
25            a_[m_-1] = y;
26            a_[m_] = z;
27            return m_;
28        }
29        else // add to one unit second left
30            // and combine last with second last
31        { // [* , X, Y, Z] --> [* , X+1, Y+Z]
32
33            if ( m_<=2 ) return 0; // current is last
34
35            a_[m_-2] += 1;
36            a_[m_-1] += z;
37            --m_;
38            return m_;
39        }
40    }
41 }

```

Computation of the successor for partitions into odd parts in subset-lex order:

```

1 // FILE: src/comb/partitions-odd-asc-subset-lex.h
2 ulong next()
3 // Loopless algorithm.
4 {
5     ulong y = a_[m_-1]; // may read sentinel a[0]
6     ulong z = a_[m_];

```

```

7
8     if ( z >= 3*y ) // can extend to the right
9     { // [* , Y, Z] --> [* , Y, Y, Y, Z-2*Y]
10        a_[m_] = y;
11        a_[m+1] = y;
12        a_[m+2] = z - 2 * y;
13        m_ += 2;
14        return m_;
15    }
16
17    if ( m_ >= n_ ) // all-ones partition
18    {
19        if ( n_ <= 5 ) return 0; // current is last
20
21        // [1, ..., 1, 1, 1, 1, 1, 1] --> [1, ..., 3, 3]
22        m_ -= 4;
23        a_[m_] = 3;
24        a_[m-1] = 3;
25        return m_;
26    }
27
28    ulong z2 = z - 2;
29    ulong y2 = y + 2;
30
31    if ( z2 >= y2 ) // add 2 units to the left
32    { // [* , Y, Z] --> [* , Y+2, Z-2]
33        a_[m-1] = y2;
34        a_[m_] = z2;
35        return m_;
36    }
37
38    if ( m_==2 ) // current is last (happens only for n even)
39        return 0;
40
41    // here m >= 3; add to second or third left:
42
43    ulong x2 = a_[m-2] + 2;
44    ulong s = z + y - 2;
45
46    if ( x2 <= z2 )
47    // add 2 units to third left, repeat part, put rest at end
48    { // [* , X, Y, Z] --> [* , X+2, X+2, Y+Z-2 -X-2]
49        a_[m-2] = x2;
50        a_[m-1] = x2;
51        a_[m_] = s - x2;
52        return m_;
53    }
54
55    if ( m_==3 ) // current is last (happens only for n odd)
56        return 0;
57
58    // add 2 units to third left, combine rest into second left
59    // [* , W, X, Y, Z] --> [* , W+2, X+Y+Z-2]
60    a_[m-3] += 2;
61    a_[m-2] += s;
62    m_ -= 2;
63    return m_;
64 }

```

The updates via `next()` of partitions into distinct parts take about 11 cycles for lexicographic order and about 9 cycles for subset-lex order. The respective figures for the partitions into odd parts are 13 and 13.5 cycles.

Algorithms for ranking and unranking are obtained by replacing 2^{n-f-1} (the number of compositions of n with first part f) in algorithms 12 and 13 by the expression for the number of partitions of n (of the desired kind) with first part f .

6 Subset-lex order for restricted growth strings (RGS)

By modifying the algorithms for the successor and predecessor (3 and 4) for subset-lex order to adhere to certain conditions, one obtains the equivalents for RGS. These can often be given without any difficulty. We give two examples, RGS for set partitions and RGS for k -ary Dyck words.

6.1 RGS for set partitions

We consider the restricted growth strings (RGS) a_0, a_1, \dots, a_{n-1} such that $a_0 = 0$ and $a_k \leq 1 + \max(a_0, a_1, \dots, a_{k-1})$. These RGS are counted by the Bell numbers, see sequence A000110 in [28]. Figure 10 shows the all such RGS of length 5 in lexicographic and subset-lex order. The set partitions are obtained by putting all i such that $a_i = k$ into the same set.

In the following algorithm we assume that $m_k = 1 + \max(a_0, a_1, \dots, a_{k-1})$ and that there is a sentinel $a_{-1} = 1$.

Algorithm 16 (Next-Setpart-RGS). *Compute the successor in subset-lex order.*

1. Let j be the index of the last nonzero digit ($j = 1$ for the all-zero RGS).
2. If $a_j < m_j$, set $a_j = a_j + 1$ and return.
3. If $j + 1 < n$, set $a_{j+1} = 1$ and return.
4. Set $a_j = 0$.
5. Set j to the position of the next nonzero digit to the left. If $j = -1$, stop.
6. Set $a_j = a_j - 1$ and $a_{j+1} = 1$.

The implementation has to take care of updating the array $m[]$ which has an additional (write-only) element at its end.

```

1 // FILE: src/comb/setpart-rgs-subset-lex.h
2 class setpart_rgs_subset_lex
3 {
4     ulong *a_; // digits of the RGS
5     ulong *m_; // maximum + 1 in prefix
6     ulong tr_; // current track
7     ulong n_; // number of digits in RGS

```

The computation of the successor is correct for all $n \geq 0$, we omit the constructor, which sets $m_0 = 1$ for $n = 0$ to cover this case.

```

1     bool next()
2     {
3         ulong j = tr_;
4         if ( a_[j] < m_[j] ) // easy case 1: can increment track
5         {
6             if ( n_ <= 1 ) return false; // handle n <= 1 correctly
7             a_[j] += 1;
8             return true;
9         }
10        const ulong j1 = j + 1;
11        if ( j1 < n_ ) // easy case 2: can attach
12        {
13            m_[j1] = m_[j] + 1;
14            a_[j1] = +1;
15            tr_ = j1;
16            return true;
17        }
18    }
19

```

	lexicographic		subset-lex	
1:	[.]	{1 2 3 4 5}	[.]	{1 2 3 4 5}
2:	[. . . . 1]	{1 2 3 4} {5}	[. . 1 . . .]	{1 3 4 5} {2}
3:	[. . . . 1 .]	{1 2 3 5} {4}	[. . 1 1 . .]	{1 4 5} {2 3}
4:	[. . . . 1 1]	{1 2 3} {4 5}	[. . 1 2 . .]	{1 4 5} {2} {3}
5:	[. . . . 1 2]	{1 2 3} {4} {5}	[. . 1 2 1 .]	{1 5} {2 4} {3}
6:	[. . . . 1 . .]	{1 2 4 5} {3}	[. . 1 2 2 .]	{1 5} {2} {3 4}
7:	[. . . . 1 . 1]	{1 2 4} {3 5}	[. . 1 2 3 .]	{1 5} {2} {3} {4}
8:	[. . . . 1 . 2]	{1 2 4} {3} {5}	[. . 1 2 3 1]	{1} {2 5} {3} {4}
9:	[. . . . 1 1 .]	{1 2 5} {3 4}	[. . 1 2 3 2]	{1} {2} {3 5} {4}
10:	[. . . . 1 1 1]	{1 2} {3 4 5}	[. . 1 2 3 3]	{1} {2} {3} {4 5}
11:	[. . . . 1 1 2]	{1 2} {3 4} {5}	[. . 1 2 3 4]	{1} {2} {3} {4} {5}
12:	[. . . . 1 2 .]	{1 2 5} {3} {4}	[. . 1 2 2 1]	{1} {2 5} {3 4}
13:	[. . . . 1 2 1]	{1 2} {3 5} {4}	[. . 1 2 2 2]	{1} {2} {3 4 5}
14:	[. . . . 1 2 2]	{1 2} {3} {4 5}	[. . 1 2 2 3]	{1} {2} {3 4} {5}
15:	[. . . . 1 2 3]	{1 2} {3} {4} {5}	[. . 1 2 1 1]	{1} {2 4 5} {3}
16:	[. . . . 1 . . .]	{1 3 4 5} {2}	[. . 1 2 1 2]	{1} {2 4} {3 5}
17:	[. . . . 1 . . 1]	{1 3 4} {2 5}	[. . 1 2 1 3]	{1} {2 4} {3} {5}
18:	[. . . . 1 . . 2]	{1 3 4} {2} {5}	[. . 1 2 . 1]	{1 4} {2 5} {3}
19:	[. . . . 1 . 1 .]	{1 3 5} {2 4}	[. . 1 2 . 2]	{1 4} {2} {3 5}
20:	[. . . . 1 . 1 1]	{1 3} {2 4 5}	[. . 1 2 . 3]	{1 4} {2} {3} {5}
21:	[. . . . 1 . 1 2]	{1 3} {2 4} {5}	[. . 1 1 1 .]	{1 5} {2 3 4}
22:	[. . . . 1 . 2 .]	{1 3 5} {2} {4}	[. . 1 1 2 .]	{1 5} {2 3} {4}
23:	[. . . . 1 . 2 1]	{1 3} {2 5} {4}	[. . 1 1 2 1]	{1} {2 3 5} {4}
24:	[. . . . 1 . 2 2]	{1 3} {2} {4 5}	[. . 1 1 2 2]	{1} {2 3} {4 5}
25:	[. . . . 1 . 2 3]	{1 3} {2} {4} {5}	[. . 1 1 2 3]	{1} {2 3} {4} {5}
26:	[. . . . 1 1 . .]	{1 4 5} {2 3}	[. . 1 1 1 1]	{1} {2 3 4 5}
27:	[. . . . 1 1 . 1]	{1 4} {2 3 5}	[. . 1 1 1 2]	{1} {2 3 4} {5}
28:	[. . . . 1 1 . 2]	{1 4} {2 3} {5}	[. . 1 1 . 1]	{1 4} {2 3 5}
29:	[. . . . 1 1 1 .]	{1 5} {2 3 4}	[. . 1 1 . 2]	{1 4} {2 3} {5}
30:	[. . . . 1 1 1 1]	{1} {2 3 4 5}	[. . 1 . 1 .]	{1 3 5} {2 4}
31:	[. . . . 1 1 1 2]	{1} {2 3 4} {5}	[. . 1 . 2 .]	{1 3 5} {2} {4}
32:	[. . . . 1 1 2 .]	{1 5} {2 3} {4}	[. . 1 . 2 1]	{1 3} {2 5} {4}
33:	[. . . . 1 1 2 1]	{1} {2 3 5} {4}	[. . 1 . 2 2]	{1 3} {2} {4 5}
34:	[. . . . 1 1 2 2]	{1} {2 3} {4 5}	[. . 1 . 2 3]	{1 3} {2} {4} {5}
35:	[. . . . 1 1 2 3]	{1} {2 3} {4} {5}	[. . 1 . 1 1]	{1 3} {2 4 5}
36:	[. . . . 1 2 . .]	{1 4 5} {2} {3}	[. . 1 . 1 2]	{1 3} {2 4} {5}
37:	[. . . . 1 2 . 1]	{1 4} {2 5} {3}	[. . 1 . . 1]	{1 3 4} {2 5}
38:	[. . . . 1 2 . 2]	{1 4} {2} {3 5}	[. . 1 . . 2]	{1 3 4} {2} {5}
39:	[. . . . 1 2 . 3]	{1 4} {2} {3} {5}	[. . . . 1 . .]	{1 2 4 5} {3}
40:	[. . . . 1 2 1 .]	{1 5} {2 4} {3}	[. . . . 1 1 .]	{1 2 5} {3 4}
41:	[. . . . 1 2 1 1]	{1} {2 4 5} {3}	[. . . . 1 1 2 .]	{1 2 5} {3} {4}
42:	[. . . . 1 2 1 2]	{1} {2 4} {3 5}	[. . . . 1 2 1]	{1 2} {3 5} {4}
43:	[. . . . 1 2 1 3]	{1} {2 4} {3} {5}	[. . . . 1 2 2]	{1 2} {3} {4 5}
44:	[. . . . 1 2 2 .]	{1 5} {2} {3 4}	[. . . . 1 2 3]	{1 2} {3} {4} {5}
45:	[. . . . 1 2 2 1]	{1} {2 5} {3 4}	[. . . . 1 1 1]	{1 2} {3 4 5}
46:	[. . . . 1 2 2 2]	{1} {2} {3 4 5}	[. . . . 1 1 2]	{1 2} {3 4} {5}
47:	[. . . . 1 2 2 3]	{1} {2} {3 4} {5}	[. . . . 1 . 1]	{1 2 4} {3 5}
48:	[. . . . 1 2 3 .]	{1 5} {2} {3} {4}	[. . . . 1 . 2]	{1 2 4} {3} {5}
49:	[. . . . 1 2 3 1]	{1} {2 5} {3} {4}	[. 1 .]	{1 2 3 5} {4}
50:	[. . . . 1 2 3 2]	{1} {2} {3 5} {4}	[. 1 1]	{1 2 3} {4 5}
51:	[. . . . 1 2 3 3]	{1} {2} {3} {4 5}	[. 1 2]	{1 2 3} {4} {5}
52:	[. . . . 1 2 3 4]	{1} {2} {3} {4} {5}	[. 1]	{1 2 3 4} {5}

Figure 10: Restricted growth strings and corresponding set partitions in lexicographic and subset-lex order.

```

20     a_[j] = 0;
21     m_[j] = m_[j-1];
22
23     // Find nonzero track to the left:
24     do { --j; } while ( a_[j] == 0 ); // can read sentinel
25
26     if ( (long)j < 0 ) return false; // current is last
27
28     if ( a_[j] == m_[j] ) m_[j+1] = m_[j];
29     a_[j] -= 1;
30
31     ++j;
32     a_[j] = 1;
33     tr_ = j;
34     return true;
35 }

```

An update takes about 9.5 cycles for subset-lex order and 12.5 cycles for lexicographic order.

6.2 RGS for k -ary Dyck words

Figure 11 shows the 55 RGS for the 3-ary Dyck words of length 12, in both lexicographic and subset-lex order. The j th value in each RGS contains the distance of the position j th one in the Dyck word from its maximal value $k \cdot j$. The sequences of numbers of k -ary Dyck words are entries A000108 ($k = 2$, Catalan numbers), A001764 ($k = 3$), A002293 ($k = 4$), and A002294 ($k = 5$) in [28].

It shall suffice to give the implementation for the (loopless) computation of the predecessor in subset-lex order. A sentinel $a[-1]=+1$ is used.

```

1 // FILE: src/comb/dyck-rgs-subset-lex.h
2 class dyck_rgs_subset_lex
3 {
4     ulong *a_; // digits of the RGS: a_[k] <= as[k-1] + 1
5     ulong tr_; // current track
6     ulong n_; // number of digits in RGS
7     ulong i_; // k-ary Dyck words: i = k - 1
8
9     void last()
10    {
11        for (ulong k=0; k<n_; ++k) a_[k] = 0;
12        tr_ = n_ - 1;
13        // make things work for n <= 1:
14        if ( n_==0 )
15        {
16            tr_ = 0;
17            a_[0] = 1;
18        }
19        if ( n_>=2 ) a_[tr_] = i_;
20    }

```

All cases $n \geq 0$ are handled correctly.

```

1     bool prev()
2     // Loopless algorithm.
3     {
4         if ( n_<=1 ) return false; // just one RGS
5
6         ulong j = tr_;
7         if ( a_[j] > 1 ) // can decrement track
8         {
9             a_[j] -= 1;
10            return true;
11        }
12
13        const ulong aj = a_[j]; // zero or one
14
15        a_[j] = 0;

```

1:	[. . .]	1..1..1..1..	[. . .]	1..1..1..1..
2:	[. . . 1]	1..1..1..1..	[. 1 . .]	1..1..1..1..
3:	[. . . 2]	1..1..11....	[. 2 . .]	11....1..1..
4:	[. . 1 .]	1..1..1..1..	[. 2 1 .]	11....1..1..
5:	[. . 1 1]	1..1..1..1..	[. 2 2 .]	11..1....1..
6:	[. . 1 2]	1..1..1..1..	[. 2 3 .]	11..1....1..
7:	[. . 1 3]	1..1..11....	[. 2 4 .]	111....1..
8:	[. . 2 .]	1..11....1..	[. 2 4 1]	111....1..
9:	[. . 2 1]	1..11....1..	[. 2 4 2]	111....1..
10:	[. . 2 2]	1..11....1..	[. 2 4 3]	111....1..
11:	[. . 2 3]	1..11....1..	[. 2 4 4]	111....1..
12:	[. . 2 4]	1..111.....	[. 2 4 5]	111..1.....
13:	[. 1 . .]	1..1..1..1..	[. 2 4 6]	1111.....
14:	[. 1 . 1]	1..1..1..1..	[. 2 3 1]	11..1....1..
15:	[. 1 . 2]	1..1..11....	[. 2 3 2]	11..1....1..
16:	[. 1 1 .]	1..1..1..1..	[. 2 3 3]	11..1....1..
17:	[. 1 1 1]	1..1..1..1..	[. 2 3 4]	11..1....1..
18:	[. 1 1 2]	1..1..1..1..	[. 2 3 5]	11..11.....
19:	[. 1 1 3]	1..1..11....	[. 2 2 1]	11..1....1..
20:	[. 1 2 .]	1..1..1..1..	[. 2 2 2]	11..1....1..
21:	[. 1 2 1]	1..1..1..1..	[. 2 2 3]	11..1....1..
22:	[. 1 2 2]	1..1..1..1..	[. 2 2 4]	11..11.....
23:	[. 1 2 3]	1..1..1..1..	[. 2 1 1]	11....1..1..
24:	[. 1 2 4]	1..1..11....	[. 2 1 2]	11....1..1..
25:	[. 1 3 .]	1..11....1..	[. 2 1 3]	11....11.....
26:	[. 1 3 1]	1..11....1..	[. 2 . 1]	11....1..1..
27:	[. 1 3 2]	1..11....1..	[. 2 . 2]	11....11.....
28:	[. 1 3 3]	1..11....1..	[. 1 1 .]	1..1..1..1..
29:	[. 1 3 4]	1..11....1..	[. 1 2 .]	1..1..1..1..
30:	[. 1 3 5]	1..111.....	[. 1 3 .]	1..11....1..
31:	[. 2 . .]	11....1..1..	[. 1 3 1]	1..11....1..
32:	[. 2 . 1]	11....1..1..	[. 1 3 2]	1..11....1..
33:	[. 2 . 2]	11....11.....	[. 1 3 3]	1..11....1..
34:	[. 2 1 .]	11....1..1..	[. 1 3 4]	1..11....1..
35:	[. 2 1 1]	11....1..1..	[. 1 3 5]	1..111.....
36:	[. 2 1 2]	11....1..1..	[. 1 2 1]	1..1..1..1..
37:	[. 2 1 3]	11....11.....	[. 1 2 2]	1..1..1..1..
38:	[. 2 2 .]	11....1..1..	[. 1 2 3]	1..1..1..1..
39:	[. 2 2 1]	11....1..1..	[. 1 2 4]	1..1..11.....
40:	[. 2 2 2]	11....1..1..	[. 1 1 1]	1..1..1..1..
41:	[. 2 2 3]	11....1..1..	[. 1 1 2]	1..1..1..1..
42:	[. 2 2 4]	11..11.....	[. 1 1 3]	1..1..11.....
43:	[. 2 3 .]	11..1..1..	[. 1 . 1]	1..1..1..1..
44:	[. 2 3 1]	11..1..1..	[. 1 . 2]	1..1..11.....
45:	[. 2 3 2]	11..1..1..	[. . 1 .]	1..1..1..1..
46:	[. 2 3 3]	11..1..1..	[. . 2 .]	1..11....1..
47:	[. 2 3 4]	11..1..1..	[. . 2 1]	1..11....1..
48:	[. 2 3 5]	11..11.....	[. . 2 2]	1..11....1..
49:	[. 2 4 .]	111....1..	[. . 2 3]	1..11....1..
50:	[. 2 4 1]	111....1..	[. . 2 4]	1..111.....
51:	[. 2 4 2]	111....1..	[. . 1 1]	1..1..1..1..
52:	[. 2 4 3]	111....1..	[. . 1 2]	1..1..1..1..
53:	[. 2 4 4]	111..1.....	[. . 1 3]	1..1..11.....
54:	[. 2 4 5]	111..1.....	[. . . 1]	1..1..1..1..
55:	[. 2 4 6]	1111.....	[. . . 2]	1..1..11.....

Figure 11: The 55 RGS for the 3-ary Dyck words of length 12, in lexicographic order (left) and subset-lex order (right).

```

16     --j;
17
18     if ( a_[j] == a_[j-1] + i_ ) // move track to the left
19     {
20         --tr_;
21         return true;
22     }
23
24     if ( j==0 ) // current or next is last
25     {
26         if ( aj == 0 ) return false;
27         return true;
28     }
29
30     a_[j] += 1; // increment left digit
31     tr_ = n_ - 1; // move to right end
32     a_[tr_] = a_[tr_-1] + i_; // set to max value
33     return true;
34 }

```

One update takes about 10 cycles for both lexicographic and subset-lex order.

For the generation of other restricted growth strings in subset-lex order, see the following files.

```

// Ascent sequences, see OEIS sequence A022493:
src/comb/ascent-rgs-subset-lex.h
src/comb/ascent-rgs.h // lexicographic order

// RGS for Catalan objects, see OEIS sequence A000108:
src/comb/catalan-rgs-subset-lex.h (loopless prev())
src/comb/catalan-rgs.h // lexicographic order

// Standard Young tableaux, represented as ballot sequences,
// see OEIS sequence A000085:
src/comb/young-tab-rgs-subset-lex.h
src/comb/young-tab-rgs.h // lexicographic order

```

7 A variant of the subset-lex order

An order obtained by processing the positions in the characteristic words as for subset-lex order but with reversed priorities is shown in figure 12. We will call this ordering *subset-lexrev*. The algorithms for computing successor and predecessor are easily obtained from those for the subset-lex order, we just give the implementation for `prev()`, which is (again) loopless.

```

1 // FILE: src/comb/mixedradix-subset-lexrev.h
2 bool prev()
3 {
4     ulong j = tr_;
5     if ( a_[j] > 1 ) // easy case: just decrement
6     {
7         a_[j] -= 1;
8         return true;
9     }
10
11     a_[j] = 0;
12     ++j; // now looking at next track to the right
13
14     if ( j >= n_ ) // was on rightmost track (last two steps)
15     {
16         bool q = ( a_[j] != 0 );
17         a_[j] = 0;
18         return q;
19     }
20
21     if ( a_[j] == m1_[j] ) // semi-easy case: move track to left
22     {
23         tr_ = j; // move track one right
24         return true;

```

0:	[. . . .]	{ }
1:	[. . . . 1]	{ 3 }
2:	[. . . . 2]	{ 3, 3 }
3:	[. . . . 3]	{ 3, 3, 3 }
4:	[. . . 1 3]	{ 3, 3, 3, 2 }
5:	[. . . 2 3]	{ 3, 3, 3, 2, 2 }
6:	[. . 1 2 3]	{ 3, 3, 3, 2, 2, 1 }
7:	[. . 2 2 3]	{ 3, 3, 3, 2, 2, 1, 1 }
8:	[1 2 2 3]	{ 3, 3, 3, 2, 2, 1, 1, 0 }
9:	[1 1 2 3]	{ 3, 3, 3, 2, 2, 1, 0 }
10:	[1 . 2 3]	{ 3, 3, 3, 2, 2, 0 }
11:	[. 1 1 3]	{ 3, 3, 3, 2, 1 }
12:	[. 2 1 3]	{ 3, 3, 3, 2, 1, 1 }
13:	[1 2 1 3]	{ 3, 3, 3, 2, 1, 1, 0 }
14:	[1 1 1 3]	{ 3, 3, 3, 2, 1, 0 }
15:	[1 . 1 3]	{ 3, 3, 3, 2, 0 }
16:	[. 1 . 3]	{ 3, 3, 3, 1 }
17:	[. 2 . 3]	{ 3, 3, 3, 1, 1 }
18:	[1 2 . 3]	{ 3, 3, 3, 1, 1, 0 }
19:	[1 1 . 3]	{ 3, 3, 3, 1, 0 }
20:	[1 . . 3]	{ 3, 3, 3, 0 }
21:	[. . . 1 2]	{ 3, 3, 2 }
22:	[. . . 2 2]	{ 3, 3, 2, 2 }
23:	[. . 1 2 2]	{ 3, 3, 2, 2, 1 }
24:	[. . 2 2 2]	{ 3, 3, 2, 2, 1, 1 }
25:	[1 2 2 2]	{ 3, 3, 2, 2, 1, 1, 0 }
26:	[1 1 2 2]	{ 3, 3, 2, 2, 1, 0 }
27:	[1 . 2 2]	{ 3, 3, 2, 2, 0 }
28:	[. 1 1 2]	{ 3, 3, 2, 1 }
29:	[. 2 1 2]	{ 3, 3, 2, 1, 1 }
30:	[1 2 1 2]	{ 3, 3, 2, 1, 1, 0 }
31:	[1 1 1 2]	{ 3, 3, 2, 1, 0 }
32:	[1 . 1 2]	{ 3, 3, 2, 0 }
33:	[. 1 . 2]	{ 3, 3, 1 }
34:	[. 2 . 2]	{ 3, 3, 1, 1 }
35:	[1 2 . 2]	{ 3, 3, 1, 1, 0 }
36:	[1 1 . 2]	{ 3, 3, 1, 0 }
37:	[1 . . 2]	{ 3, 3, 0 }
38:	[. . . 1 1]	{ 3, 2 }
39:	[. . . 2 1]	{ 3, 2, 2 }
40:	[. . 1 2 1]	{ 3, 2, 2, 1 }
41:	[. . 2 2 1]	{ 3, 2, 2, 1, 1 }
42:	[1 2 2 1]	{ 3, 2, 2, 1, 1, 0 }
43:	[1 1 2 1]	{ 3, 2, 2, 1, 0 }
44:	[1 . 2 1]	{ 3, 2, 2, 0 }
45:	[. 1 1 1]	{ 3, 2, 1 }
46:	[. 2 1 1]	{ 3, 2, 1, 1 }
47:	[1 2 1 1]	{ 3, 2, 1, 1, 0 }
48:	[1 1 1 1]	{ 3, 2, 1, 0 }
49:	[1 . 1 1]	{ 3, 2, 0 }
50:	[. 1 . 1]	{ 3, 1 }
51:	[. 2 . 1]	{ 3, 1, 1 }
52:	[1 2 . 1]	{ 3, 1, 1, 0 }
53:	[1 1 . 1]	{ 3, 1, 0 }
54:	[1 . . 1]	{ 3, 0 }
55:	[. . . 1 .]	{ 2 }
56:	[. . . 2 .]	{ 2, 2 }
57:	[. . 1 2 .]	{ 2, 2, 1 }
58:	[. . 2 2 .]	{ 2, 2, 1, 1 }
59:	[1 2 2 .]	{ 2, 2, 1, 1, 0 }
60:	[1 1 2 .]	{ 2, 2, 1, 0 }
61:	[1 . 2 .]	{ 2, 2, 0 }
62:	[. 1 1 .]	{ 2, 1 }
63:	[. 2 1 .]	{ 2, 1, 1 }
64:	[1 2 1 .]	{ 2, 1, 1, 0 }
65:	[1 1 1 .]	{ 2, 1, 0 }
66:	[1 . 1 .]	{ 2, 0 }
67:	[. 1 . .]	{ 1 }
68:	[. 2 . .]	{ 1, 1 }
69:	[1 2 . .]	{ 1, 1, 0 }
70:	[1 1 . .]	{ 1, 0 }
71:	[1 . . .]	{ 0 }

Figure 12: Subsets of the set $\{0^1, 1^2, 2^2, 3^3\}$ in subset-lexrev order. Dots denote zeros in the (generalized) characteristic words. Note the sets are printed starting with the largest element.

	lex	colex	subset-lexrev	subset-lex
1:	[.]	[.]	[.]	[.]
2:	[. . . . 1]	[. . . . 1]	[. . . . 1]	[. 1 2 3 3]
3:	[. . . . 2]	[. . . . 1 1]	[. . . . 2]	[. 1 2 3 4]
4:	[. . . . 3]	[. . . 1 1 1]	[. . . . 3]	[. 1 2 2 2]
5:	[. . . . 4]	[. . 1 1 1 1]	[. . . . 4]	[. 1 2 2 3]
6:	[. . . . 1 1]	[. 2]	[. . . . 1 4]	[. 1 2 2 4]
7:	[. . . . 1 2]	[. . . . 1 2]	[. . . . 2 4]	[. 1 1 3 3]
8:	[. . . . 1 3]	[. . . . 1 1 2]	[. . . . 3 4]	[. 1 1 3 4]
9:	[. . . . 1 4]	[. . 1 1 1 2]	[. . . 1 3 4]	[. 1 1 2 2]
10:	[. . . . 2 2]	[. . . . 2 2]	[. . . 2 3 4]	[. 1 1 2 3]
11:	[. . . . 2 3]	[. . . 1 2 2 2]	[. . 1 2 3 4]	[. 1 1 2 4]
12:	[. . . . 2 4]	[. . 1 1 2 2 2]	[. 1 1 3 4]	[. 1 1 1 1]
13:	[. . . . 3 3]	[. . . 2 2 2 2]	[. . . 1 2 4]	[. 1 1 1 2]
14:	[. . . . 3 4]	[. . 1 2 2 2 2]	[. . . 2 2 4]	[. 1 1 1 3]
15:	[. . . 1 1 1 1]	[. 3]	[. 1 2 2 4]	[. 1 1 1 4]
16:	[. . . 1 1 2]	[. . . . 1 3]	[. 1 1 2 4]	[. . . 2 3 3]
17:	[. . . 1 1 3]	[. . . . 1 1 3]	[. . . 1 1 4]	[. . . 2 3 4]
18:	[. . . 1 1 4]	[. . 1 1 1 3]	[. 1 1 1 4]	[. . . 2 2 2]
19:	[. . . 1 2 2]	[. . . . 2 3]	[. . . . 1 3]	[. . . 2 2 3]
20:	[. . . 1 2 3]	[. . . 1 2 3 3]	[. . . . 2 3]	[. . . 2 2 4]
21:	[. . . 1 2 4]	[. . 1 1 2 3 3]	[. . . . 3 3]	[. . . 1 3 3]
22:	[. . . 1 3 3]	[. . . 2 2 3 3]	[. . . 1 3 3]	[. . . 1 3 4]
23:	[. . . 1 3 4]	[. . 1 2 2 3 3]	[. . . 2 3 3]	[. . . 1 2 2]
24:	[. . . 2 2 2]	[. . . . 3 3 3]	[. 1 2 3 3]	[. . . 1 2 3]
25:	[. . . 2 2 3]	[. . . 1 3 3 3]	[. 1 1 3 3]	[. . . 1 2 4]
26:	[. . . 2 2 4]	[. . 1 1 3 3 3]	[. . . 1 2 3]	[. . . 1 1 1]
27:	[. . . 2 3 3]	[. . . 2 3 3 3]	[. . . 2 2 3]	[. . . 1 1 2]
28:	[. . . 2 3 4]	[. . 1 2 3 3 3]	[. 1 2 2 3]	[. . . 1 1 3]
29:	[. . 1 1 1 1]	[. 4]	[. 1 1 2 3]	[. . . 1 1 4]
30:	[. . 1 1 1 2]	[. . . . 1 4]	[. . 1 1 3]	[. . . . 3 3]
31:	[. . 1 1 1 3]	[. . . 1 1 4]	[. 1 1 1 3]	[. . . . 3 4]
32:	[. . 1 1 1 4]	[. . 1 1 1 4]	[. . . . 1 2]	[. . . . 2 2]
33:	[. . 1 1 2 2]	[. . . . 2 4]	[. . . . 2 2]	[. . . . 2 3]
34:	[. . 1 1 2 3]	[. . . 1 2 4]	[. . . 1 2 2]	[. . . . 2 4]
35:	[. . 1 1 2 4]	[. . 1 1 2 4]	[. . . 2 2 2]	[. . . . 1 1]
36:	[. . 1 1 3 3]	[. . . 2 2 4]	[. 1 2 2 2]	[. . . . 1 2]
37:	[. . 1 1 3 4]	[. . 1 2 2 4]	[. 1 1 2 2]	[. . . . 1 3]
38:	[. . 1 2 2 2]	[. . . . 3 4]	[. . . 1 1 2]	[. . . . 1 4]
39:	[. . 1 2 2 3]	[. . . 1 3 4]	[. 1 1 1 2]	[. 1]
40:	[. . 1 2 2 4]	[. . 1 1 3 4]	[. . . . 1 1]	[. 2]
41:	[. . 1 2 3 3]	[. . . 2 3 4]	[. . . 1 1 1]	[. 3]
42:	[. . 1 2 3 4]	[. . 1 2 3 4]	[. 1 1 1 1]	[. 4]

Figure 13: RGS corresponding to certain lattice paths (see text) in lexicographic, co-lexicographic, subset-lexrev, and subset-lex order.

```

25     }
26     else
27     {
28         a_[j] += 1; // increment digit to the right
29         j = 0;
30         a_[j] = m1_[j]; // set leftmost digit = nine
31         tr_ = j; // move to leftmost track
32         return true;
33     }
34 }

```

One use of the subset-lexrev order is the development of algorithms for the generation of orderings for objects where the subset-lex order exhibits no simple structure.

As an example we consider the restricted growth strings corresponding to lattice paths from $(0, 0)$ to (n, n) with steps $(+1, 0)$ and $(+1, +1)$ that do not go below the diagonal. These RGS are words a_0, a_1, \dots, a_{n-1} such that $a_0 = 0, a_j \leq j$, and $a_{j+1} \geq a_j$ (the final $a_n = n$ is omitted). These RGS are counted by the Catalan numbers, see sequence A000108 in [28].

Figure 13 shows the all such RGS of length 5 in lexicographic, co-lexicographic, subset-lexrev, and subset-lex order. The listing in subset-lex order does not seem to have any apparently useful features, while the listing in subset-lexrev order suggests the

following method of generation.

We now assume a sentinel $a_n = +\infty$ at the end of the word.

Algorithm 17 (Next-Catalan-Step-RGS). *Compute the successor in subset-lexrev order.*

1. Let a_t be the first nonzero digit.
2. If $a_t < a_{t+1}$ and $a_t < t$, increment a_t and return.
3. If $t \geq 2$, set $a_{t-1} = 1$ and return.
4. Remove all leading ones (setting them to zero), let j be the position of the first digit $a_j \neq 1$.
5. If the word is all-zero (that is, $j = n$), stop.
6. Set $a_j = a_j - 1$ and set $a_{j-1} = 1$.

The implementation correctly handles all cases $n \geq 0$.

```

1 // FILE: src/comb/catalan-step-rgs-subset-lexrev.h
2 class catalan_step_rgs_subset_lexrev
3 {
4     ulong *a_; // RGS
5     ulong n2_; // aux: min(n,2).
6     ulong tr_; // aux: track we are looking at
7     ulong n_; // length of RGS

```

The routine for the successor returns the position of the rightmost change.

```

1     ulong next()
2     {
3         const ulong a0 = a_[tr_];
4
5         if ( a0 < a_[tr_+1] ) // may read sentinel
6         {
7             if ( a0 < tr_ ) // can increment
8             {
9                 a_[tr_] = a0 + 1;
10                return tr_;
11            }
12        }
13
14        if ( tr_ != 1 ) // can move left and increment (from 0 to 1)
15        {
16            --tr_;
17            a_[tr_] = 1;
18            return tr_;
19        }
20
21        // remove ones:
22        ulong j = tr_;
23        do { a_[j] = 0; } while ( a_[++j] == 1 );
24
25        if ( j==n2_ ) return 0; // current was last
26
27        // decrement first value != 1:
28        ulong aj = a_[j] - 1;
29        a_[j] = aj;
30
31        // move left and restore the 1:
32        tr_ = j - 1;
33        a_[tr_] = 1;
34        return j; // rightmost change at j==tr+1
35    }

```

An update takes about 10.5 cycles, whereas the updates for lexicographic and co-lexicographic order respectively take 9 and 8 cycles.

We mention that the subset-lexrev order coincides for certain objects with well-known

0:	[1 1 1 1 1 1 1 1 1 1]	[1 1 1 1 1 1 1 1 1 1]
1:	[1 1 1 1 1 1 1 1 2 .]	[2 1 1 1 1 1 1 1 1 .]
2:	[1 1 1 1 1 1 1 3 . .]	[2 2 1 1 1 1 1 1 . .]
3:	[1 1 1 1 1 1 2 2 . .]	[3 1 1 1 1 1 1 1 . .]
4:	[1 1 1 1 1 1 4 . . .]	[2 2 2 1 1 1 1 . . .]
5:	[1 1 1 1 1 2 3 . . .]	[3 2 1 1 1 1 1 . . .]
6:	[1 1 1 1 2 2 2 . . .]	[4 1 1 1 1 1 1 . . .]
7:	[1 1 1 1 5]	[2 2 2 2 1 1]
8:	[1 1 1 2 4]	[3 2 2 1 1 1]
9:	[1 1 1 3 3]	[3 3 1 1 1 1]
10:	[1 1 1 2 2 3]	[4 2 1 1 1 1]
11:	[1 1 2 2 2 2]	[5 1 1 1 1 1]
12:	[1 1 1 1 6]	[2 2 2 2 2]
13:	[1 1 1 2 5]	[3 2 2 2 1]
14:	[1 1 1 3 4]	[3 3 2 1 1]
15:	[1 1 2 2 4]	[4 2 2 1 1]
16:	[1 1 2 3 3]	[4 3 1 1 1]
17:	[1 2 2 2 3]	[5 2 1 1 1]
18:	[2 2 2 2 2]	[6 1 1 1 1]
19:	[1 1 1 7]	[3 3 2 2]
20:	[1 1 2 6]	[4 2 2 2]
21:	[1 1 3 5]	[3 3 3 1]
22:	[1 2 2 5]	[4 3 2 1]
23:	[1 1 4 4]	[5 2 2 1]
24:	[1 2 3 4]	[4 4 1 1]
25:	[2 2 2 4]	[5 3 1 1]
26:	[1 3 3 3]	[6 2 1 1]
27:	[2 2 3 3]	[7 1 1 1]
28:	[1 1 8]	[4 3 3]
29:	[1 2 7]	[4 4 2]
30:	[1 3 6]	[5 3 2]
31:	[2 2 6]	[6 2 2]
32:	[1 4 5]	[5 4 1]
33:	[2 3 5]	[6 3 1]
34:	[2 4 4]	[7 2 1]
35:	[3 3 4]	[8 1 1]
36:	[1 9]	[5 5]
37:	[2 8]	[6 4]
38:	[3 7]	[7 3]
39:	[4 6]	[8 2]
40:	[5 5]	[9 1]
41:	[10]	[10]

Figure 14: The partitions of 10 in subset-lexrev order, as weakly increasing lists of parts (left) and as weakly decreasing lists of parts (right).

orderings. For example, for partitions as lists of parts as shown in figure 14, both orderings are by falling length of partitions as major order, the minor order is lexicographic in case of weakly increasing parts and reverse co-lexicographic for weakly decreasing parts.

Similar observations can be made (for example) for compositions into a fixed number of parts, for both subset-lex and subset-lexrev order.

8 SL-Gray order for binary words

A well-known construction for the binary reflected Gray code proceeds by successively reversing ranges having identical prefixes that end in a one, using increasingly longer prefixes, see top of figure 15.

The same construction, now using prefixes ending in a zero, gives a Gray code for subset-lex order, see bottom of figure 15. We will call this ordering the *SL-Gray order*.

For the computation of the successor we keep a variable t for the current track and a variable d indicating the direction in which the track will be moved if necessary. Initially $t = 0$ and $d = +1$.

Algorithm 18 (Next-SL-Gray). *Compute the successor in SL-Gray order.*

1. If $d = +1$, do the following (try to append trailing ones):
 - (a) If $a_t = 0$, set $a_t = 1$, $t = t + 1$, and return.
 - (b) Otherwise, set $d = -1$ (change direction), $t = n - 1$ (move to rightmost track), $j = n - 2$, $a_j = 1 - a_j$, and return.
2. Otherwise ($d = -1$), do the following (try to remove trailing ones):
 - (a) If $a_{t-1} = 1$, set $a_t = 0$, $t = t - 1$, and return.
 - (b) Otherwise, set $d = +1$ (change direction), $j = t - 2$, $a_j = 1 - a_j$, $t = t + 1$ (move right), and return.

The algorithm is loopless. In the implementation, the variables `tr` and `dt` respectively correspond to t and d in the algorithm. Two sentinels $a_{-1} = a_n = +1$ are used.

```

1 // FILE: src/comb/binary-sl-gray.h
2 class binary_sl_gray
3 {
4     ulong n_; // number of digits
5     ulong tr_; // aux: current track (0 <= tr <= n)
6     ulong dt_; // aux: direction in which track tries to move
7     ulong *a_; // digits
8
9     void first()
10    {
11        for (ulong k=0; k<n_; ++k) a_[k] = 0;
12        tr_ = 0;
13        dt_ = +1;
14    }
15
16    explicit binary_sl_gray(ulong n)
17    {
18        n_ = n;
19        a_ = new ulong[n_+2]; // sentinels at both ends
20
21        a_[n_+1] = +1; // != 0
22        a_[0] = +1;
23
24        ++a_; // nota bene
25

```

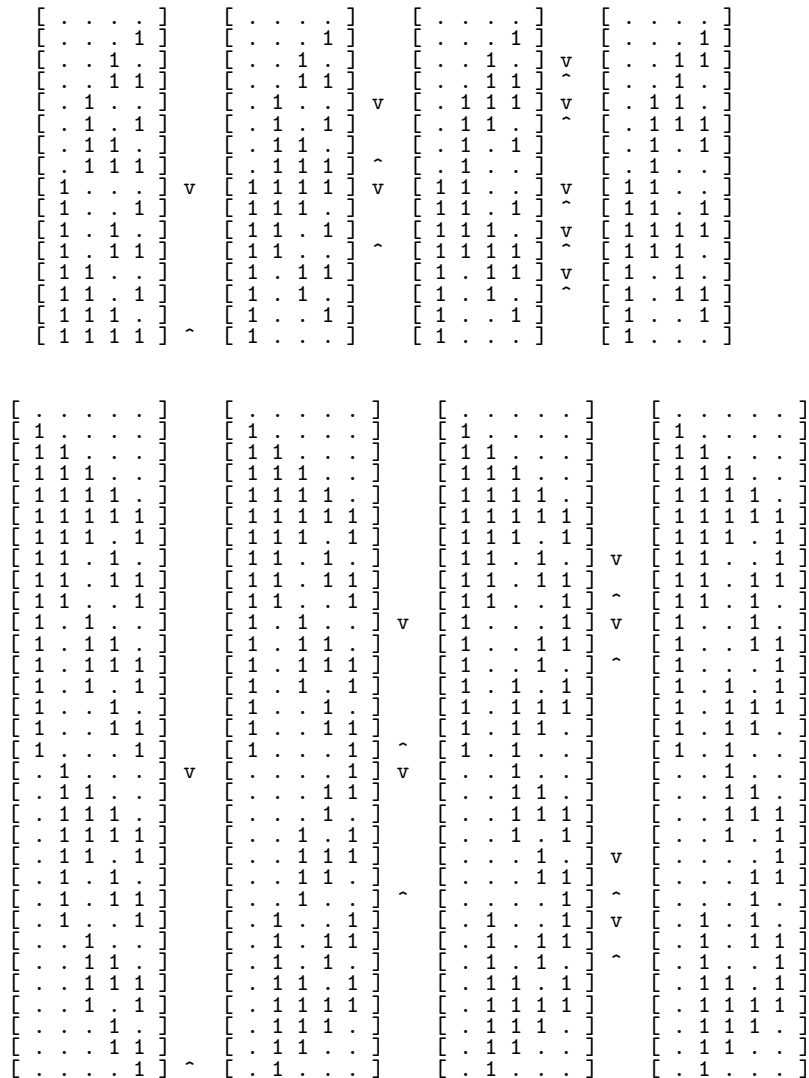


Figure 15: Construction of Gray codes by reversing sublists: for the binary reflected Gray code (top) and for the binary SL-Gray order (bottom). The symbols v and ^ respectively mark begin and end of the reversed sublists.

```

26     first();
27 }
28
29 void first()
30 {
31     for (ulong k=0; k<n_; ++k) a_[k] = 0;
32     tr_ = 0;
33     dt_ = +1;
34
35     j_ = ( n_>=2 ? 1 : 0); // wrt. last word
36     dm_ = -1; // wrt. last word
37 }
38

```

The routines for successor and predecessor handle all cases $n \geq 0$ correctly.

```

1     bool next()
2     {
3         if ( dt_ == +1 ) // try to append trailing ones
4         {
5             if ( a_[tr_] == 0 ) // can append // may read sentinel a[n]
6             {
7                 a_[tr_] = 1;
8                 ++tr_;
9             }
10            else
11            {
12                dt_ = -1UL; // change direction
13                tr_ = n_ - 1;
14                ulong j_ = tr_ - 1;
15                // Is current last (only for n_ <= 1)?
16                if ( j_ > n_ ) return false;
17                a_[j_] = 1 - a_[j_];
18            }
19        }
20        else // dt_ == -1 // try to remove trailing ones
21        {
22            if ( a_[tr_-1] != 0 ) // can remove // tr - 1 >= 0
23            {
24                a_[tr_] = 0;
25                --tr_;
26            }
27            else
28            {
29                dt_ = +1; // change direction
30                ulong j_ = tr_ - 2;
31                if ( (long)j_ < 0 ) return false;
32                a_[j_] = 1 - a_[j_];
33                ++tr_;
34            }
35        }
36        return true;
37    }
38

```

The routine to compute the predecessor is obtained by (essentially) negating the direction the track tries to move (variable dt):

```

1     bool prev()
2     {
3         if ( dt_ != +1 ) // dt== -1 // try to append trailing ones
4         {
5             if ( a_[tr_+1] == 0 ) // can append // may read sentinel a[n]
6             {
7                 a_[tr_+1] = 1;
8                 ++tr_;
9             }
10            else
11            {
12                dt_ = +1; // change direction
13                tr_ = n_ - 1;
14                ulong j_ = tr_ - 1;
15                a_[j_] = 1 - a_[j_];
16                ++tr_;
17            }

```

```

18     }
19     else // dt_ == +1 // try to remove trailing ones
20     {
21         if ( tr_ == 0 )
22         {
23             if ( a_[0]==0 ) return false; // (only for n_ <= 1)
24             a_[0] = 0;
25             return true;
26         }
27
28         // tr - 1 >= -1 (can read low sentinel)
29         if ( a_[tr_-2] != 0 ) // can remove
30         {
31             a_[tr_-1] = 0;
32             --tr_;
33         }
34         else
35         {
36             dt_ = -1UL; // change direction
37             ulong j_ = tr_ - 3;
38             a_[j_] = 1 - a_[j_];
39             --tr_;
40         }
41     }
42     return true;
43 }
44

```

One update in either direction takes about 7.5 cycles. An implementation for the generation of the SL-Gray order in a binary word is given in `src/bits/bit-sl-gray.h`.

The ranking algorithm is easily obtained by observing that if the highest bit is not set (and the word is nonzero), the order for the remaining word is reflected, as shown in figure 15.

Algorithm 19 (Binary-SL-Gray-Rank). *Recursive routine $F(k, n)$ for the computation of the rank of an n -bit word k in binary SL-Gray order.*

1. If $k = 0$, return 0.
2. Set $w = k$ and unset the highest bit in w .
3. If the highest bit of k (at position $n - 1$) is set, return $1 + F(w, n - 1)$.
4. Otherwise return $2^n - F(w, n - 1)$.

In the following implementation the variable `ldn` must give the number of bits in the SL-Gray code.

```

1 // FILE: src/bits/bin-to-sl-gray.h
2 ulong sl_gray_to_bin(ulong k, ulong ldn)
3 {
4     if ( k==0 ) return 0;
5     ulong b = 1UL << (ldn-1); // mask for bit at end
6     ulong h = k & b; // bit at end
7     k ^= h;
8     ulong z = sl_gray_to_bin( k, ldn-1 ); // recursion
9     if ( h==0 ) return (b<<1) - z;
10    else return 1 + z;
11 }

```

A routine for the conversion of a binary word into the corresponding word in SL-Gray order (unranking algorithm) is

```

1 ulong bin_to_sl_gray(ulong k, ulong ldn)
2 {
3     if ( ldn==0 ) return 0;
4
5     ulong b = 1UL << (ldn-1); // highest bit
6     ulong m = (b<<1) - 1; // mask for reversing direction
7     ulong z = b; // Gray code
8     k -= 1; // move all-zero word to begin

```

```

0121030121014101210301210125210121030121014101210301210123
.....1...1...1...1...1...1...1...1...1...1...1...1...1...1...1...
.....1...1...1...1...1...1...1...1...1...1...1...1...1...1...1...
.....1...1...1...1...1...1...1...1...1...1...1...1...1...1...1...
.....1...1...1...1...1...1...1...1...1...1...1...1...1...1...1...
.....1...1...1...1...1...1...1...1...1...1...1...1...1...1...1...
.....1111..1111...1111..1111.....1111..1111...1111..1111...
.....11..1111..11..11..1111..11...11..1111..11..11..1111..11...
.....1111.....11111111.....1111..1111.....11111111.....1111...
.....11111111.....1111111111111111.....11111111.....11111111...
.....1111111111111111.....1111111111111111.....1111111111111111
.....1111111111111111111111111111.....1111111111111111.....

```

Figure 16: The delta sequence (top) of the binary SL-Gray order, starting after the initial slope and indexing positions from the end of the words.

```

9      while ( b != 0 )
10     {
11         const along h = k & b; // bit under consideration
12         z ^= h; // with one, switch bit in Gray code
13         if ( !h ) k ^= m; // reverse direction with zero
14         k += 1; // SL-Gray
15         b >>= 1; // next lower bit
16         m >>= 1; // next smaller mask
17     }
18     return z;
19 }
20 }

```

The delta sequence (sequence of positions of changes), starting after the initial slope and indexing positions from the end, is shown in figure 16, this is sequence A217262 in [28]. It can be obtained from the ruler function (sequence A007814) by replacing 0 by $w = 01210$, 1 by 3, 2 by 141, 3 by 12521, 4 by 1236321, ..., n by $123\dots(n-1)(n+2)(n-1)\dots321$:

```

0 1 0 2 0 1 0 3 0 1 0 2 0 1 0 4 0 ... (ruler function)
w 3 w 141 w 3 w 12521 w 3 w 141 w 3 w 1236321 w ... =
01210 3 01210 141 01210 3 01210 12521 01210 3 01210 141 01210 3 01210 ...

```

It is easy to see that (for $n \geq 4$) the SL-Gray order for the 2^n words of length n has $2^{n-2} - 2$ successive transitions that are 3-close (have a distance of 3), the rest are 1-close (adjacent changes).

A Gray code where the maximal distance between successive transitions is k is called k -close. In [2, p.400] it has been observed that 1-close Gray codes exist for $n \leq 6$ but not for $n = 7$ (and it appears unlikely that for any $n \geq 8$ such a Gray code exists). Does a 2-close Gray code exist? We remark that the SL-Gray order is 2-close if the radices for all digits are even and ≥ 4 .

9 SL-Gray order for mixed radix words

The generalization of the SL-Gray order to mixed radix words is obtained by successive reversion of the sublists with identical prefix for all prefixes ending in an even digit.

For the computation of the successor we keep a variable t for the current track and variables $d_j = \pm 1$ (an array) for the direction the digit a_j is currently moving in (whether its is incremented or decremented). We further use m_j for the maximal value the digit a_j can have (the “nines”). Initially all digits a_k are zero, all directions d_k are +1, and $t = 0$.

0:	[. . . .]	[+ + + +]	{ }
1:	[1 . . .]	[+ + + +]	{ 0 }
2:	[1 1 . .]	[- + + +]	{ 0, 1 }
3:	[1 2 . .]	[- + + +]	{ 0, 1, 1 }
4:	[1 2 1 .]	[- - + +]	{ 0, 1, 1, 2 }
5:	[1 2 2 .]	[- - + +]	{ 0, 1, 1, 2, 2 }
6:	[1 2 2 1]	[- - - +]	{ 0, 1, 1, 2, 2, 3 }
7:	[1 2 2 2]	[- - - +]	{ 0, 1, 1, 2, 2, 3, 3 }
8:	[1 2 2 3]	[- - - +]	{ 0, 1, 1, 2, 2, 3, 3, 3 }
9:	[1 2 1 3]	[- - - -]	{ 0, 1, 1, 2, 3, 3, 3 }
10:	[1 2 1 2]	[- - - -]	{ 0, 1, 1, 2, 3, 3 }
11:	[1 2 1 1]	[- - - -]	{ 0, 1, 1, 2, 3 }
12:	[1 2 . 1]	[- - - +]	{ 0, 1, 1, 3 }
13:	[1 2 . 2]	[- - - +]	{ 0, 1, 1, 3, 3 }
14:	[1 2 . 3]	[- - - +]	{ 0, 1, 1, 3, 3, 3 }
15:	[1 1 . 3]	[- - + -]	{ 0, 1, 3, 3, 3 }
16:	[1 1 . 2]	[- - + -]	{ 0, 1, 3, 3 }
17:	[1 1 . 1]	[- - + -]	{ 0, 1, 3 }
18:	[1 1 1 1]	[- - + +]	{ 0, 1, 2, 3 }
19:	[1 1 1 2]	[- - + +]	{ 0, 1, 2, 3, 3 }
20:	[1 1 1 3]	[- - + +]	{ 0, 1, 2, 3, 3, 3 }
21:	[1 1 2 3]	[- - + -]	{ 0, 1, 2, 2, 3, 3, 3 }
22:	[1 1 2 2]	[- - + -]	{ 0, 1, 2, 2, 3, 3 }
23:	[1 1 2 1]	[- - + -]	{ 0, 1, 2, 2, 3 }
24:	[1 1 2 .]	[- - - +]	{ 0, 1, 2, 2 }
25:	[1 1 1 .]	[- - - +]	{ 0, 1, 2 }
26:	[1 . 1 .]	[- - + +]	{ 0, 2 }
27:	[1 . 2 .]	[- - + +]	{ 0, 2, 2 }
28:	[1 . 2 1]	[- - + +]	{ 0, 2, 2, 3 }
29:	[1 . 2 2]	[- - + +]	{ 0, 2, 2, 3, 3 }
30:	[1 . 2 3]	[- - + +]	{ 0, 2, 2, 3, 3, 3 }
31:	[1 . 1 3]	[- - - -]	{ 0, 2, 3, 3, 3 }
32:	[1 . 1 2]	[- - - -]	{ 0, 2, 3, 3 }
33:	[1 . 1 1]	[- - - -]	{ 0, 2, 3 }
34:	[1 . . 1]	[- - - +]	{ 0, 3 }
35:	[1 . . 2]	[- - - +]	{ 0, 3, 3 }
36:	[1 . . 3]	[- - - +]	{ 0, 3, 3, 3 }
37:	[. . . 3]	[- + + -]	{ 3, 3, 3 }
38:	[. . . 2]	[- + + -]	{ 3, 3 }
39:	[. . . 1]	[- + + -]	{ 3 }
40:	[. . . 1 1]	[- + + +]	{ 2, 3 }
41:	[. . . 1 2]	[- + + +]	{ 2, 3, 3 }
42:	[. . . 1 3]	[- + + +]	{ 2, 3, 3, 3 }
43:	[. . 2 3]	[- + + -]	{ 2, 2, 3, 3, 3 }
44:	[. . 2 2]	[- + + -]	{ 2, 2, 3, 3 }
45:	[. . 2 1]	[- + + -]	{ 2, 2, 3 }
46:	[. . 2 .]	[- + - +]	{ 2, 2 }
47:	[. . 1 .]	[- + - +]	{ 2 }
48:	[. 1 1 .]	[- + + +]	{ 1, 2 }
49:	[. 1 2 .]	[- + + +]	{ 1, 2, 2 }
50:	[. 1 2 1]	[- + - +]	{ 1, 2, 2, 3 }
51:	[. 1 2 2]	[- + - +]	{ 1, 2, 2, 3, 3 }
52:	[. 1 2 3]	[- + - +]	{ 1, 2, 2, 3, 3, 3 }
53:	[. 1 1 3]	[- + - -]	{ 1, 2, 3, 3, 3 }
54:	[. 1 1 2]	[- + - -]	{ 1, 2, 3, 3 }
55:	[. 1 1 1]	[- + - -]	{ 1, 2, 3 }
56:	[. 1 . 1]	[- + - +]	{ 1, 3 }
57:	[. 1 . 2]	[- + - +]	{ 1, 3, 3 }
58:	[. 1 . 3]	[- + - +]	{ 1, 3, 3, 3 }
59:	[. 2 . 3]	[- + + -]	{ 1, 1, 3, 3, 3 }
60:	[. 2 . 2]	[- + + -]	{ 1, 1, 3, 3 }
61:	[. 2 . 1]	[- + + -]	{ 1, 1, 3 }
62:	[. 2 1 1]	[- + + +]	{ 1, 1, 2, 3 }
63:	[. 2 1 2]	[- + + +]	{ 1, 1, 2, 3, 3 }
64:	[. 2 1 3]	[- + + +]	{ 1, 1, 2, 3, 3, 3 }
65:	[. 2 2 3]	[- + + -]	{ 1, 1, 2, 2, 3, 3, 3 }
66:	[. 2 2 2]	[- + + -]	{ 1, 1, 2, 2, 3, 3 }
67:	[. 2 2 1]	[- + + -]	{ 1, 1, 2, 2, 3 }
68:	[. 2 2 .]	[- + - +]	{ 1, 1, 2, 2 }
69:	[. 2 1 .]	[- + - +]	{ 1, 1, 2 }
70:	[. 2 . .]	[- - + +]	{ 1, 1 }
71:	[. 1 . .]	[- - + +]	{ 1 }

Figure 17: Mixed radix words (left) and corresponding subsets (right) of the multiset $\{0^1, 1^2, 2^2, 3^3\}$ in SL-Gray order, and the array of directions used in the computation (middle).

Algorithm 20 (Next-SL-Gray). *Compute the successor SL-Gray order.*

1. Set $j = t$ and $b = a_j + d_j$.
2. If $b \neq 0$ and $b \leq m_j$, set $a_j = b$ and return (easy case).
3. Set $d_j = -d_j$ (change direction for digit j).
4. If $d_j = +1$ and $a_{j+1} = 0$, set $a_{j+1} = +1$, $t = j + 1$ (move track right), and return.
5. If $d_j = -1$ and $a_{j-1} = m_{j-1}$, set $a_j = 0$, $d_{j-1} = -1$, $t = j - 1$ (move track left), and return.
6. Find the position p of the nearest digit a_k to the left such that $a_k + d_k$ is in the range $0, 1, \dots, m_k$ (a valid digit). In the process, change d_k for all k where $p < k < j$.
7. Set $a_p = a_p + d_p$ (change digit, keep track).

The details for termination have been omitted, this is handled with the help of sentinels in the implementation.

```

1 // FILE: src/comb/mixedradix-sl-gray.h
2 class mixedradix_sl_gray
3 {
4     ulong n_; // number of digits
5     // (n kinds of elements in multiset, n>=1)
6     ulong tr_; // aux: current track
7     ulong *a_; // digits of mixed radix number
8     // (multiplicity of kind k in subset).
9     ulong *d_; // directions (either +1 or -1)
10    ulong *m1_; // nines (radix minus one) for each digit
11    // (multiplicity of kind k in set).
```

Sentinels are used in all arrays at index -1 and at index n to handle termination.

```

1 mixedradix_sl_gray(ulong n, ulong mm, const ulong *m=0)
2 // Must have n>=1
3 {
4     n_ = n;
5     a_ = new ulong[n_+2]; // all with sentinels at both ends
6     d_ = new ulong[n_+2];
7     m1_ = new ulong[n_+2];
8
9     // sentinels on the right:
10    a_[n_+1] = +1; // != 0
11    m1_[n_+1] = +1; // same as a_[n+1]
12    d_[n_+1] = +1; // positive
13
14    // sentinels on the left:
15    a_[0] = +2; // >= +2
16    m1_[0] = +2; // same as a_[0]
17    d_[0] = 0; // zero
18
19    ++a_; ++d_; ++m1_; // nota bene
20
21    mixedradix_init(n_, mm, m, m1_);
22    first();
23 }
24
25 void first()
26 {
27     for (ulong k=0; k<n_; ++k) a_[k] = 0;
28     for (ulong k=0; k<n_; ++k) d_[k] = +1;
29     tr_ = 0;
30 }
```

The method for the successor handles all cases $n \geq 0$ correctly. The variable `a1` in the implementation corresponds to b in the algorithm.

```

1  bool next()
2  {
3      ulong j = tr_;
4      const ulong dj = d_[j];
5      const ulong a1 = a_[j] + dj; // a[j] +- 1
6
7      if ( (a1 != 0) && (a1 <= m1_[j]) ) // easy case
8      {
9          a_[j] = a1;
10         return true;
11     }
12
13     d_[j] = -dj; // change direction
14
15     if ( dj == +1 ) // so a_[j] == m1_[j] == nine
16     {
17         // Try to move track right with a[j] == nine:
18         const ulong j1 = j + 1;
19         if ( a_[j1] == 0 ) // can read high sentinel
20         {
21             a_[j1] = +1;
22             tr_ = j1;
23             return true;
24         }
25     }
26     else // here dj == -1, so a_[j] == 1
27     {
28         if ( (long)j <= 0 ) return false; // current is last
29
30         // Try to move track left with a[j] == 1:
31         const ulong j1 = j - 1;
32         if ( a_[j1] == m1_[j1] ) // can read low sentinel when n_ == 1
33         {
34             a_[j] = 0;
35             d_[j1] = -1UL;
36             tr_ = j1;
37             return true;
38         }
39     }
40
41     // find first changeable track to the left:
42     --j;
43     while ( a_[j] + d_[j] > m1_[j] ) // may read low sentinels
44     {
45         d_[j] = -d_[j]; // change direction
46         --j;
47     }
48
49     if ( (long)j < 0 ) return false; // current is last
50
51     // Change digit left but keep track:
52     a_[j] += d_[j];
53
54     return true;
55 }

```

10 A Gray code for compositions

We now describe a Gray code for compositions where at most one unit is moved in each step, all moves are 1-close or 2-close (these always cross a part 1), and all moves are at the end of the current composition.

The compositions of 7 in this ordering are shown in figure 18 (left), it can be obtained from the lexicographic order by successively reversing the sublists with identical prefixes that end in an odd part. To keep matters simple for the iterative algorithm, we arrange the compositions for n odd such that the first part is only decreasing (starting with the composition $[n]$) and otherwise such that the first part is increasing (starting

0:	111111	[7]	0:	111.....	[4 1 1 1 1 1]
1:	11111.	[6 1]	1:	11..1...	[3 1 2 1 1 1]
2:	1111..	[5 1 1]	2:	11....1.	[3 1 1 1 2 1]
3:	1111.1	[5 2]	3:	11....1	[3 1 1 1 1 2]
4:	111.11	[4 3]	4:	11...1..	[3 1 1 2 1 1]
5:	111.1.	[4 2 1]	5:	11.1....	[3 2 1 1 1 1]
6:	111...	[4 1 1 1]	6:	1.11....	[2 3 1 1 1 1]
7:	111..1	[4 1 2]	7:	1.1.1...	[2 2 2 1 1 1]
8:	11..11	[3 1 3]	8:	1.1...1.	[2 2 1 1 2 1]
9:	11...1.	[3 1 2 1]	9:	1.1....1	[2 2 1 1 1 2]
10:	11....	[3 1 1 1 1]	10:	1.1..1..	[2 2 1 2 1 1]
11:	11...1	[3 1 1 2]	11:	1...11..	[2 1 1 3 1 1]
12:	11.1..	[3 2 1 1]	12:	1...1.1.	[2 1 1 2 2 1]
13:	11.1.1	[3 2 2]	13:	1...1..1	[2 1 1 2 1 2]
14:	11.11.	[3 3 1]	14:	1....11	[2 1 1 1 1 3]
15:	11.111	[3 4]	15:	1...1.1	[2 1 1 1 2 2]
16:	1.1111	[2 5]	16:	1...11.	[2 1 1 1 3 1]
17:	1.111.	[2 4 1]	17:	1..1..1.	[2 1 2 1 2 1]
18:	1.11..	[2 3 1 1]	18:	1..1...1	[2 1 2 1 1 2]
19:	1.11.1	[2 3 2]	19:	1..1.1..	[2 1 2 2 1 1]
20:	1.1.11	[2 2 3]	20:	1..11...	[2 1 3 1 1 1]
21:	1.1.1.	[2 2 2 1]	21:	..111...	[1 1 4 1 1 1]
22:	1.1...	[2 2 1 1 1]	22:	..11..1.	[1 1 3 1 2 1]
23:	1.1..1	[2 2 1 2]	23:	..11...1	[1 1 3 1 1 2]
24:	1...11	[2 1 1 3]	24:	..11.1..	[1 1 3 2 1 1]
25:	1...1.	[2 1 1 2 1]	25:	..1.11..	[1 1 2 3 1 1]
26:	1.....	[2 1 1 1 1 1]	26:	..1.1.1.	[1 1 2 2 2 1]
27:	1...1	[2 1 1 1 2]	27:	..1.1..1	[1 1 2 2 1 2]
28:	1..1..	[2 1 2 1 1]	28:	..1...11	[1 1 2 1 1 3]
29:	1..1.1	[2 1 2 2]	29:	..1..1.1	[1 1 2 1 2 2]
30:	1..11.	[2 1 3 1]	30:	..1..11.	[1 1 2 1 3 1]
31:	1..111	[2 1 4]	31:111.	[1 1 1 1 4 1]
32:	..1111	[1 1 5]	32:11.1	[1 1 1 1 3 2]
33:	..111.	[1 1 4 1]	33:1.11	[1 1 1 1 2 3]
34:	..11..	[1 1 3 1 1]	34:111	[1 1 1 1 1 4]
35:	..11.1	[1 1 3 2]	35:	...1..11	[1 1 1 2 1 3]
36:	..1.11	[1 1 2 3]	36:	...1.1.1	[1 1 1 2 2 2]
37:	..1.1.	[1 1 2 2 1]	37:	...1.11.	[1 1 1 2 3 1]
38:	..1...	[1 1 2 1 1 1]	38:	...11.1.	[1 1 1 3 2 1]
39:	..1..1	[1 1 2 1 2]	39:	...11..1	[1 1 1 3 1 2]
40:	...11	[1 1 1 1 3]	40:	...111..	[1 1 1 4 1 1]
41:1.	[1 1 1 1 2 1]	41:	.1..11..	[1 2 1 3 1 1]
42:	[1 1 1 1 1 1 1]	42:	.1..1.1.	[1 2 1 2 2 1]
43:1	[1 1 1 1 1 2]	43:	.1..1..1	[1 2 1 2 1 2]
44:	...1..	[1 1 1 1 2 1 1]	44:	.1....11	[1 2 1 1 1 3]
45:	...1.1	[1 1 1 1 2 2]	45:	.1...1.1	[1 2 1 1 2 2]
46:	...1.1	[1 1 1 1 3 1]	46:	.1...11.	[1 2 1 1 3 1]
47:	...111	[1 1 1 1 4]	47:	.1.1..1.	[1 2 2 1 2 1]
48:	.1..11	[1 2 1 3]	48:	.1.1...1	[1 2 2 1 1 2]
49:	.1..1.	[1 2 1 2 1]	49:	.1.1.1..	[1 2 2 2 1 1]
50:	.1....	[1 2 1 1 1 1]	50:	.1.11...	[1 2 3 1 1 1]
51:	.1...1	[1 2 1 1 2]	51:	.11.1...	[1 3 2 1 1 1]
52:	.1.1..	[1 2 2 1 1]	52:	.11...1.	[1 3 1 1 2 1]
53:	.1.1.1	[1 2 2 2]	53:	.11....1	[1 3 1 1 1 2]
54:	.1.11.	[1 2 3 1]	54:	.11..1..	[1 3 1 2 1 1]
55:	.1.111	[1 2 4]	55:	.111....	[1 4 1 1 1 1]
56:	.11.11	[1 3 3]			
57:	.11.1.	[1 3 2 1]			
58:	.11...	[1 3 1 1 1]			
59:	.11..1	[1 3 1 2]			
60:	.111..	[1 4 1 1]			
61:	.111.1	[1 4 2]			
62:	.1111.	[1 5 1]			
63:	.11111	[1 6]			

Figure 18: The 64 compositions of 7 together with their binary encodings as in figure 6 (left) and the 56 compositions of 9 into exactly 6 parts (right).

with the composition $[1, n - 1]$. The compositions of 6 in this ordering are obtained by dropping the first part 1 in all compositions of 7 in the second half of the list, see figure 18.

A recursive routine can be obtained by switching between the recursive functions for lexicographic and reversed lexicographic order whenever an odd part has been written. In the following a global array $a[]$ is used.

```

1 // FILE: src/comb/composition-nz-gray-rec-demo.cc
2 void F(ulong n, ulong m)
3 {
4     if ( n==0 ) { visit(m); return; }
5     for (ulong f=n; f!=0; --f) // first part decreasing
6     {
7         a[m] = f;
8         if ( 0 == (f & 1) ) F(n-f, m+1);
9         else B(n-f, m+1);
10    }
11 }
12
13 void B(ulong n, ulong m)
14 {
15     if ( n==0 ) { visit(m); return; }
16     for (ulong f=1; f<=n; ++f) // first part increasing
17     {
18         a[m] = f;
19         if ( 0 == (f & 1) ) B(n-f, m+1);
20         else F(n-f, m+1);
21     }
22 }
23 }
24

```

The initial call is $F(n, 0)$ for n odd and $G(n, 0)$ for n even.

For the iterative algorithm we use a function $D(x)$ that shall return $+1$ if x is odd and otherwise -1 . We will refer to the last three parts as respectively x , y , and z .

Algorithm 21 (Next-Comp-Gray). *Compute the successor of a composition.*

1. If $z = n - 1$ and n is odd, or $z = n$ and n is even, stop.
2. If $z = 1$, do the following:
 - (a) If $D(y) = +1$, set $y = y + 1$ and drop z ; return.
 - (b) Otherwise ($D(y) = -1$), set $y = y - 1$ and append a part 1; return.
3. Otherwise ($z > 1$) do the following:
 - (a) If z is odd, set $z = z - 1$ and append a part 1; return.
 - (b) If $y > 1$, set $y = y - D(y)$ and $z = z + D(y)$; return.
 - (c) If $x > 1$, set $x = x + D(x)$ and $z = z - D(x)$; return.
 - (d) Otherwise ($x = 1$) set $x = x + 1$ and $z = z - 1$.

The algorithm is loopless. Note that the initialization is loopless as well.

The implementation handles all $n \geq$ correctly.

```

1 // FILE: src/comb/composition-nz-gray2.h
2 class composition_nz_gray2
3 {
4     ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // compositions of n
6     ulong m_; // current composition has m parts
7     ulong e_; // aux: detection of last composition
8
9     explicit composition_nz_gray2(ulong n)
10    {

```

```

11     n_ = n;
12     a_ = new ulong[n_+1+(n_==0)];
13     a_[0] = 0; // returned by last_part() when n==0
14     a_[1] = 0; // returned by first_part() when n==0
15
16     if ( n_ <= 1 ) e_ = n_;
17     else          e_ = ( oddq(n_) ? n_ - 1 : n_ );
18
19     first();
20 }
21
22 void first()
23 {
24     if ( n_ <= 1 )
25     {
26         a_[1] = n_;
27         m_ = n_;
28     }
29     else
30     {
31         if ( oddq(n_) )
32         {
33             a_[1] = n_;
34             m_ = 1;
35         }
36         else
37         {
38             a_[1] = 1;
39             a_[2] = n_ - 1;
40             m_ = 2;
41         }
42     }
43 }
44

```

A few auxiliary routines are used.

```

1     bool oddq(ulong x) const { return 0 != ( x & 1UL ); }
2     bool evenq(ulong x) const { return 0 == ( x & 1UL ); }
3
4     ulong par_to_dir_odd(ulong x) const
5     {
6         if ( oddq(x) ) return +1;
7         else          return -1UL;
8     }
9
10    ulong par_to_dir_even(ulong x) const
11    {
12        if ( evenq(x) ) return +1;
13        else          return -1UL;
14    }

```

We split the update into routines for the cases $z == 1$ and $z > 1$ as in the algorithm.

```

1     ulong next_zeq1() // for Z == 1
2     {
3         const ulong y = a_[m_-1];
4         const ulong dy = par_to_dir_odd(y);
5
6         if ( dy == +1 )
7         { // [*, Y, 1 ] --> [*, Y+1 ]
8             a_[m_-1] = y + 1;
9             m_ -= 1;
10            return m_;
11        }
12        else // dy == -1
13        { // [*, Y, 1 ] --> [*, Y-1, 1, 1 ]
14            a_[m_-1] = y - 1;
15            m_ += 1;
16            a_[m_] = 1;
17            return m_;
18        }
19    }
20
21    ulong next_zgt1() // for Z > 1
22    {

```

```

23     const ulong z = a_[m_];
24     const ulong y = a_[m_-1];
25
26     if ( oddq(z) )
27     { // [* , Z ] --> [* , Z-1 , 1 ]
28         a_[m_] = z - 1;
29         m_ += 1;
30         a_[m_] = 1;
31         return m_;
32     }
33
34     if ( y != 1 ) // Y > 1
35     { // [* , Y , Z ] --> [* , Y+-1 , Z+-1 ]
36         const ulong dy = par_to_dir_even(y);
37         a_[m_-1] = y + dy;
38         a_[m_] = z - dy;
39         return m_;
40     }
41     else // Y == 1
42     {
43         const ulong x = a_[m_-2];
44
45         if ( x != 1 )
46         { // [* , X , 1 , Z ] --> [* , X+-1 , 1 , Z+-1 ]
47             const ulong dx = par_to_dir_odd(x);
48             a_[m_-2] = x + dx;
49             a_[m_] = z - dx;
50             return m_;
51         }
52         else // X == 1
53         { // [* , X , 1 , Z ] --> [* , X+1 , 1 , Z-1 ]
54             a_[m_-2] = x + 1;
55             a_[m_] = z - 1;
56             return m_;
57         }
58     }
59 }

```

The routine `next()` returns number of parts in the new composition and zero if there are no more compositions.

```

1     ulong next()
2     {
3         ulong z = a_[m_];
4         if ( z == e_ ) return 0; // current is last
5
6         if ( z != 1 ) return next_zgt1();
7         else return next_zeq1();
8     }

```

One update takes about 11 cycles.

We remark that the sublists of compositions into a fixed number of parts correspond to the combinations in *enup* order. The list of binary words on the right of figure 18 are those shown in [2, fig.6.6-B (left), p.189].

Ranking and unranking

Recursive routines for ranking and unranking are obtained easily. The following implementations are for the ordering with the first part decreasing.

```

1 // FILE: src/comb/composition-nz-rank.cc
2 ulong
3 composition_nz_gray_rank(const ulong *x, ulong m, ulong n)
4 // Return rank r of composition x[], 0 <= r < 2**(n-1)
5 // where n is the sum of all parts.
6 {
7     if ( m <= 1 ) return 0;
8
9     ulong f = x[0]; // first part

```

1:	1.....	{ 0 }			(continued)
2:	11....	{ 0, 1 }	23:	.1.11.	{ 1, 3, 4 }
3:	111...	{ 0, 1, 2 }	24:	.1.1.1	{ 1, 3, 5 }
4:	11.1..	{ 0, 1, 3 }	25:	.1..1.	{ 1, 4 }
5:	11..1.	{ 0, 1, 4 }	26:	.1..11	{ 1, 4, 5 }
6:	11...1	{ 0, 1, 5 }	27:	.1...1	{ 1, 5 }
7:	1.1...	{ 0, 2 }	28:	..1...	{ 2 }
8:	1.11..	{ 0, 2, 3 }	29:	..11..	{ 2, 3 }
9:	1.1.1.	{ 0, 2, 4 }	30:	..111.	{ 2, 3, 4 }
10:	1.1..1	{ 0, 2, 5 }	31:	..11.1	{ 2, 3, 5 }
11:	1..1..	{ 0, 3 }	32:	..1.1.	{ 2, 4 }
12:	1..11.	{ 0, 3, 4 }	33:	..1.11	{ 2, 4, 5 }
13:	1..1.1	{ 0, 3, 5 }	34:	..1..1	{ 2, 5 }
14:	1...1.	{ 0, 4 }	35:	...1..	{ 3 }
15:	1...11	{ 0, 4, 5 }	36:	...11.	{ 3, 4 }
16:	1....1	{ 0, 5 }	37:	...111	{ 3, 4, 5 }
17:	.1....	{ 1 }	38:	...1.1	{ 3, 5 }
18:	.11...	{ 1, 2 }	39:1.	{ 4 }
19:	.111..	{ 1, 2, 3 }	40:11	{ 4, 5 }
20:	.11.1.	{ 1, 2, 4 }	41:1	{ 5 }
21:	.11..1	{ 1, 2, 5 }			
22:	.1.1..	{ 1, 3 }			

Figure 19: Nonempty subsets of the set $\{0, 1, 2, \dots, 5\}$ with at most 3 elements.

```

10     ulong s = n - f; // remaining sum
11
12     ulong y = composition_nz_gray_rank(x+1, m-1, s);
13     if ( 0 == ( f & 1 ) ) // first part even
14         return ( 1UL << (s-1) ) + y;
15     else
16         return ( 1UL << s ) - 1 - y;
17 }
18
19     ulong
20 composition_nz_gray_unrank(ulong r, ulong *x, ulong n)
21 // Generate composition x[] of n with rank r.
22 // Return number of parts m of generated composition, 0 <= m <= n.
23 {
24     if ( r == 0 )
25     {
26         if ( n==0 ) return 0;
27         x[0] = n;
28         return 1;
29     }
30
31     ulong h = highest_one_idx(r);
32     ulong f = n - 1 - h; // first part
33     x[0] = f;
34
35     ulong b = 1UL << h; // highest one
36     r ^= b; // delete highest one
37
38     bool p = f & 1; // first part f odd ?
39     if ( p ) r = b - 1 - r; // change direction with odd f
40
41     return 1 + composition_nz_gray_unrank( r , x+1, n-f );
42 }

```

11 Appendix: nonempty subsets with at most k elements

The modifications needed in algorithm 1 for restricting the subsets to those with a prescribed maximal number of elements are quite small. Without ado, we give the implementation.

```
1 // FILE: src/comb/ksubset-lex.h
```



```

2  class ksubset_lex
3  {
4      ulong n_; // number of elements in set, should have n>=1
5      ulong j_; // number of elements in subset
6      ulong m_; // max number of elements in subsets
7      ulong *x_; // x[0...j-1]: subset of {0,1,2,...,n-1}

```

The computation of the successor is

```

1      ulong next()
2      {
3          ulong j1 = j_ - 1;
4          ulong z1 = x_[j1] + 1;
5          if ( z1 < n_ ) // last element is not max
6          {
7              if ( j_ < m_ ) // append element
8              {
9                  x_[j_] = x_[j1] + 1;
10                 ++j_;
11                 return j_;
12             }
13             x_[j1] = z1; // increment last element
14             return j_;
15         }
16         else // last element is max
17         {
18             if ( j1 == 0 ) return 0; // current is last
19             --j_;
20             x_[j_-1] += 1;
21             return j_;
22         }
23     }
24 }
25

```

We omit the routine for the predecessor. Updates take no more time than the updates for the unrestricted subsets.

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