Edge-covering plane-filling curves on grid colorings: a pedestrian approach

Jörg Arndt, Julia Handl

December 4, 2023

Abstract

We describe families of plane-filling curves on any edge-to-edge tiling of the plane with regular polygons and finitely many classes of edges. It is shown how to partition the minimal number of edge classes from the group G of symmetries of the tiling into refined colorings of the tiling, corresponding to finite subgroups of G. All of these colorings correspond to families of plane-filling curves which we call curve-sets. Our exposition is driven by illustrated examples.

Contents

1	Ter	Terminology and definitions						
	1.1	$1 k$ -uniform grids and their edge classes $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$						
	1.2	2 Edge-covering curves and their tiles						
	1.3	3 Curves on grids with more than one edge class: curve-sets						
		1.3.1 All edge classes in a patch have the same cardinality	7					
		1.3.2 The prototiles of a grid	8					
	1.4	Refined grid colorings	10					
		1.4.1 $$ A first example: a coloring of the square grid with two colors $$.	10					
		1.4.2 Dekking's tile criteria	12					
		1.4.3 Creating refined grid colorings	15					
2	Exa	amples of curve-sets	17					
	2.1	The square grid	17					
		2.1.1 The minimal coloring with one color	17					
		2.1.2 A coloring with two colors	17					
		2.1.3 The other coloring with two colors	20					
		2.1.4 A coloring with four colors	20					
		2.1.5 Another coloring with four colors	22					
	2.2	The triangle grid	25					
		2.2.1 The minimal coloring with one color	25					
		2.2.2 A coloring with three colors	26					
		2.2.3 Another coloring with three colors	28					
		2.2.4 Yet another coloring with three colors	32					
		2.2.5 A coloring with four colors	36					
	2.3	The trihexagonal grid	37					
		2.3.1 The minimal coloring with one color	37					
		2.3.2 A coloring with two colors	38					
		2.3.3 A coloring with three colors	41					
		2.3.4 A coloring with six colors	43					
	2.4	The $(3.4.6.4)$ -grid	45					
		2.4.1 The minimal coloring with two colors	45					
		2.4.2 A coloring with four colors	50					
		2.4.3 A coloring with six colors	53					
	2.5	The $(3^6; 3.3.6.6)$ -grid	54					
	2.6 The $(3.4.4.6; 3.4.6.4)$ -grid							
3	Cur	ve-sets for grids with odd valencies	63					
	3.1	The hexagon grid	64					
	3.2	The square grid	69					
	3.3	The triangle grid	73					
	3.4	The tribexagonal grid	79					
	3.5	The (4.8.8)-grid	81					
	3.6	The (3.12.12)-grid	84					
4	Cur	we-sets not fitting into our framework	87					
T	4.1	First example	87					
	4.2	Second example	88					
T	da	of topping	00					
ın	uex	or terms	90					

1 Terminology and definitions

We fix some terminology.

1.1 *k*-uniform grids and their edge classes



Figure 1.1.0-A: Tessellations of the plane with their vertex configurations, from left to right: square tiling (4^4) , triangle tiling (3^6) , trihexagonal tiling (3.6.3.6), and the tiling (3.4.6.4).

An edge-to-edge tiling (also called tessellation or grid) by regular polygons is commonly defined by specifying the classes of points (vertices). Four grids with just one class of points are shown in Figure 1.1.0-A, the respective symbols (4^4) , (3^6) , (3.6.3.6), and (3.4.6.4) specify the numbers of edges of the polygons adjacent to each point. Among the possible cyclic rotations the lexicographic minimum is chosen.

A grid with two point classes is shown in Figure 1.1.0-B. The symmetries of the grid usually allowed are translations, rotations, and reflections. A tiling with k classes of points is called k-uniform, see [11, Section 2.2, pp. 65ff].



Figure 1.1.0-B: The (3.4.4.6; 3.4.6.4)-grid (left) and its CCW (middle) and CW (right) prototiles.

For our purposes, we are rather interested in the *edge classes*. Moreover, we disallow reflections and use directed edges. We use one color and the equivalent letter for each edge class in the grid drawing.

We use the term *grid* also to refer to the *directed* grids from here on.

A grid can be defined using the *transitions* between successive edges. The transitions are triples (F, t, G) with (edge-)letters F and G and turn t have the property that the pair (F, t) uniquely defines G. We call this the *unique transition property*. In other words, the first two entries in a transition determine the third. It is also true that the last two entries determine the first. In our (3.4.4.6; 3.4.6.4)-grid shown in Figure 1.1.0-B the transitions can be given as

A++A, B++++C, C++++D, D++++B, E+++F, F+++E

 $edge\ class$

grid

transitions

unique transition property

1.2 Edge-covering curves and their tiles

A----B, B----F, C---E, D----A, E---C, F----D

where a word of j letters + or - respectively is a counterclockwise or clockwise turn by $j 2\pi/12 = j \cdot 30$ degrees. We will use the abbreviations CW and CCW respectively for clockwise and counterclockwise sense of rotation from this point on.

Another way of defining a grid is by specifying the *prototiles*. The prototiles of the (3.4.4.6; 3.4.6.4)-grid are shown on the right in Figure 1.1.0-B. We denote the CCW tiles of this grid by $[A++]^6$, $[B+++C+++D++++]^1$, and $[E+++F+++]^2$ and the CW tiles by $[A--B--F--D--]^1$ and $[C--E--]^3$. The exponent at the right indicates a repetition, for example, $[E+++F+++]^2$ could also be written as [E+++F+++E+++F+++]. Note that a prototile can appear in more then one orientation. For example, note how the prototile $[A--B--F--D---]^1$ (top right in Figure 1.1.0-B) appears in six orientations around the blue hexagons.

A *lattice tile* is a tile that tiles the plane in such a way that it appears in just one *l* orientation. For brevity we sometimes just say *tile* instead of prototile, (whether or *t* not it is a lattice tile). This will never lead to confusion, if a prototile in a grid coloring is a lattice tile, we say so.



Figure 1.1.0-C: Directed grids, from left to right: square grid (4^4) , triangle grid (3^6) , trihexagonal grid (3.6.3.6), and the (3.4.6.4)-grid.

If there is just one CW (or CCW) tile and it appears in just one orientation, then it is a lattice tile: it can be decomposed into smaller copies of itself, each of the same orientation; repeating this process shows that it is indeed a lattice tile. This happens for both tiles of the square (4^4) grid and the triangle (3^6) grid, and for the hexagonal tile of the trihexagonal grid (3.6.3.6), see Figure 1.1.0-C. The triangular tile of the trihexagonal grid appears in two orientations, therefore is not a lattice tile.

We'll give examples where certain arrangements of several prototiles give a lattice tile.

1.2 Edge-covering curves and their tiles

For the description of curves on a grid we use Lindenmayer systems as described in [2, Section 1.2, pp. 3ff]. A *Lindenmayer system* (or L-system) is a triple (Ω, A, P) where Ω is an alphabet, A a word over Ω (called the *axiom*), and P a set of maps from letters $\in \Omega$ to words over Ω that contains one map for each letter.

The word that a letter is mapped to is called the *production* of the letter. We also use the term *map* for this. If the production of a letter is just the letter itself, we call the letter a *constant*.

We specify curves by L-systems interpreted as a sequence of unit-length edges and turns. The curves can be rendered via turtle graphics, see [17, Section 1.3, pp. 6ff]. The initial direction is arbitrary. Letters are interpreted as "draw a unit stroke in the current direction", + and - as turns by \pm a fixed angle $\varphi = \frac{2\pi}{n}$ where $n \in \mathbb{N}_+$. We will also use the letter 0 for turns by 0° (non-turns).

Applying the maps of the L-system j times to the axiom gives a word that is the jth iterate. We call the drawing corresponding to this word the jth iterate of the curve. iterate

CW sense of rotation CCW sense of rotation prototile

lattice tile tile

Lindenmayer system axiom

production map constant

1.2 Edge-covering curves and their tiles

The first, second, and fourth iterates for the axiom F (corresponding to a single edge) with production $F \mapsto FOF+FOFOF-F+F-F-FOF+F+F-F$ are given in Figure 1.2.0-A. The first iterate is also called the motif, the motif for a letter F corresponds to the drawing motif of the production of F. The letters + and - respectively denote turns by ± 120 degrees, and 0 for a (non-)turn by 0 degrees. The curve is drawn with rounded corners to make the sequence of strokes apparent.



Figure 1.2.0-A: First iterate (left), second iterate (middle), and fourth iterate (right) of a curve corresponding to the word FOF+FOFOF-F+F-F-FOF+F+F-F. This curve lives on the triangle grid. Coloring by edges of the motif.

Some sources, for example [14] and [15], call the axiom the initiator and use the term generator for the (single) map of a Lindenmayer system with only one non-constant symbol.

The colors used in Figure 1.2.0-A do not correspond to edge classes; the curve lives on the triangle grid where there is just one class. Instead, each stroke of the motif has its own (arbitrary) color and the parts of the higher iterates of the curve inherit the color of the edge it developed from. This makes the *self-similarity* of the curve obvious, each colored part in the drawing on the right has the same shape, and magnifying those shapes gives the shape of the entire drawing.

All of our curves will be *self-avoiding* in the sense that both self-crossings and doubly drawn edges are forbidden.

We call a (self-avoiding) curve *edge-covering* if a large enough iterate traverses all edges in an arbitrarily large disc on the grid. Such a curve is also *plane-filling*. There are plane-filling curves of other types as well, for example, curves that traverse all points in such a disc, well-known examples being the Hilbert curve and the Peano curve.

Figure 1.2.0-B shows a rendering of the curve already seen in Figure 1.2.0-A where each edge is drawn as a lozenge stretching to the midpoint of the neighboring triangles. We call this an *area drawing* (for the lack of a better term). Area drawings are useful for emphasizing the shape of a curve and we will use it for many images without saying so. Indeed, the fourth iterate on the right in Figure 1.2.0-A is an area drawing. The idea for drawing curves this way appeared in [7, Addendum, pp.603-605].

The number of edges incident to a point (in either direction) will be called the *valency* of the point. It appears that each point needs to have an even valency, because for every stroke to the point, a stroke away from the point is needed. We will, however, solve this apparent problem in Section 3 on page 63.

self-similarity

self-avoiding

edge-covering plane-filling

area drawing

valency



Figure 1.2.0-B: First (left), second (middle), and third (right) iterate of a curve, rendered as area drawings.

1.3 Curves on grids with more than one edge class: curve-sets

Edge-covering curves on grids with just one edge class have been treated in [2]. For grids with j > 1 edge classes we have j curves working together to fill the plane, we call this a *curve-set*.

curve-set



Figure 1.3.0-A: The minimal coloring of the (3.4.6.4)-grid (left). Its CCW prototiles $[A^{++}]^6$ and $[B^{++++}]^3$ (middle), and the CW prototile $[A^{---}B^{---}]^2$ (right).

The two curves of the curve-set have the maps

The first iterate of both curves are shown in Figure 1.3.0-B.

The curves of a curve-set exhibit *mutual self-similarity* as shown in Figure 1.3.0-C: *mutual self-similarity*

both curves are decomposed into smaller copies of each other. The decomposition into colors is done similarly as in Figure 1.2.0-A on page 5.



Figure 1.3.0-B: First iterates (motifs) of a curve-set on the (3.4.6.4)-grid. Motif for A (left) and for B (right).



Figure 1.3.0-C: Fourth iterates of the curve-set, for A (left) and B (right). Coloring by orientation of the curves.

The substitution matrix of a curve-set is the matrix whose entry r, c is the number of substitution matrix letters r in the production of class c. For this curve-set the matrix is

13	6]
12	7

It tells us that curve A (left in Figure 1.3.0-C) is decomposed into 13 smaller copies of curve A (itself) and 12 smaller copies of curve B (left column of the matrix). Curve B (right in Figure 1.3.0-C) is decomposed into 6 copies of curve A and 7 copies of curve B (itself). This can be seen in Figure 1.3.0-B as well.

We observe that all row sums R of the matrix are equal (R = 19). This allows us to define the *order* of a curve-set to be R. The definition of the order in [2, bottom of *order* p.4] is a special case of this definition.

1.3.1 All edge classes in a patch have the same cardinality

We show that the row sums of the substitution matrix are indeed always identical, hence our concept of order is well-defined.

Consider a parallelogram (minimal or not) satisfying the translational symmetries of the grid. We call such a parallelogram composed of $r \times s$ minimal parallelograms an



Figure 1.3.1-A: (2,3)-patches on the square grid, separated by the diagonal light blue lines. Each of the 24 edges in a patch has a unique color and equivalent letter.

(r, s)-patch. Figure 1.3.1-A shows (2, 3)-patches on the square grid. A patch has the patch same rotational symmetries as the grid. The group of the translational symmetries of the patch is finite; it is the direct product of two cyclic groups of orders r and s.

Let G be the group of all symmetries of the edges for a patch. Consider the group action $G \times X \to X$ where X is the set of the edges in the patch. Observe that no element in G apart from the identity fixes any edge. This is trivially true for translations, but also true for rotations as we consider *directed* edges and disallow reflections. The set of group elements fixing an element (edge) x is called the stabilizer of x, denoted as $\operatorname{Stab}(x)$. We just observed that the stabilizer of all elements contains just one element, the neutral element of the group, so $|\operatorname{Stab}(x)| = 1$.

Burnside's lemma (see [13, Theorem 25.17, p. 293-294]) tells us that

$$|\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)| = |G| \qquad \forall x \in X$$

where $|\operatorname{Orb}(x)|$ is the number of edges equivalent to the edge x and |G| the number of elements of G. This is called the Orbit-Stabilizer Theorem, see [1, Theorem (17.2) and Corollary (17.3), p. 94]. Now $|\operatorname{Stab}(x)| = 1$, so

$$|\operatorname{Orb}(x)| = |G| \qquad \forall x \in X$$

In other words, all equivalence classes have the same size, namely |G|. Note that our argument falls apart when either using undirected edges or including reflections in the group.

1.3.2 The prototiles of a grid

The prototiles of a grid fall into two classes, some have CCW sense of rotation (tiles $[A++]^6$ and $[B++++]^3$ in Figure 1.3.0-A on page 6), others have CW sense (tile $[A---B---]^2$ in Figure 1.3.0-A). The first iterates (with respect to our curve-set) of the CCW prototiles are shown in Figure 1.3.2-A.

In the limit, each prototile obtains a shape determined by the curve-set, see Figure 1.3.2-B. These shapes are mutually self-similar: the shape of the tile $[A++]^6$ is that of the smaller blue regions, the shape of $[B++++]^3$ is that of the red regions. The two CCW prototiles tile the plane as shown in Figure 1.3.2-C.



Figure 1.3.2-A: First iterates of the prototiles $[A++]^6$ (left) and $[B++++]^3$ (right).



Figure 1.3.2-B: Shapes of the CCW prototiles $[\tt A++]^6$ (left) and $[\tt B++++]^3$ (right), note the mutual self-similarity.



Figure 1.3.2-C: Together, the two CCW prototiles tile the plane.

There is just one CW prototile, shown in Figure 1.3.2-D. Its shape is self-similar, it can be decomposed into 19 smaller copies of itself (colored by orientation, right in figure). The CW prototile tiles the plane, occurring in three orientations.



Figure 1.3.2-D: First iterate (left) and shape (right) of the CW prototile [A---B---]².

What we observed here is always true. All prototiles of one sense of rotation (CW or CCW) are mutually self-similar. Also, all prototiles of one sense of rotation together tile the plane.

The exponent in the notation for a prototile is the guaranteed rotational symmetry of the shape of the tile. For example, the tile $[B^{++++}]^3$ will have at least 3-fold rotational symmetry. Sometimes greater symmetries are possible, as we will show later. For brevity, we sometimes call a k-fold rotational symmetry just k-fold symmetry.

1.4 Refined grid colorings

The grid colorings considered so far correspond to the full group G of the symmetries of the grid. These are the *minimal colorings* of the grid, containing the least number of classes. Using a finite subgroup of G one obtains *refined colorings*, containing more classes.

One can always put each edge in a patch into its own class. Figure 1.3.1-A on page 8 shows such a maximally refined coloring for (2,3)-patches on the square grid. The maximally refined coloring on the square grid was used in [4, pp. 132–134] for (1,1)-patches and [4, pp. 136–138] for (1,2)-patches.

1.4.1 A first example: a coloring of the square grid with two colors

As an example of a refined coloring we use the square grid. We reduce the 4-fold rotational symmetry of the square grid to a 2-fold symmetry to obtain the coloring shown in Figure 1.4.1-A. We assign the letters L and R respectively to the horizontal and vertical edges. The prototiles are $[L+R+]^2$ and $[L-R-]^2$. The transitions for this coloring are L+R, L-R, R+L, and R-L.

The first iterates of a curve-set of order 13 with maps $L \mapsto L-R-L+R-L+R+L+R-L+R-L-R+L$ and $R \mapsto R+L-R-L-R+L+R+L-R-L-R+L+R$ are shown in Figure 1.4.1-B. The decompositions of the curves for L and R into smaller copies of themselves are shown in Figure 1.4.1-C.

k-fold symmetry

minimal coloring refined coloring



Figure 1.4.1-A: Coloring of the square grid with two colors (left) and its two prototiles $[L+R+]^2$ (middle) and $[L-R-]^2$ (right).



Figure 1.4.1-B: First iterates for a curve-set of order 13: left for the letter L, right for R. Coloring by orientation of the edges.



Figure 1.4.1-C: Mutual self-similarity of the two curves: each can be decomposed into smaller copies of both curves. Coloring by orientation of the small copies.

Dekking treated *folding curves* on this coloring of the square grid [9]. For those *folding curves* curve-sets the production of R is obtained from the production of L by reversing it and swapping turns + with - and letters L with R. In other words, the curve for R is that of L, in reversed order. The curve we just presented is not a folding curve.



Figure 1.4.1-D: First iterates of a folding curve of order 9, curve for L (left) and curve for R (right). Note that either is the reversal of the other. Coloring by letters.

The first iterates of a folding curve of order 9 with maps $L \mapsto L+R+L-R+L+R-L+R-L$ and $R \mapsto R+L-R+L-R-L+R-L-R$ are shown in Figure 1.4.1-D. The shapes of either curve (both L or R) and either tile ($[L+R+]^2$ or $[L-R-]^2$) are shown in Figure 1.4.1-E. That the shapes of both curves coincide is a consequence of being folding curves.



Figure 1.4.1-E: Left: shape of either curve (L or R), right: shape of either tile $([L+R+]^2 \text{ or } [L-R-]^2)$.



Figure 1.4.1-F: The tile $[L+R+]^2$, colored by orientation of the curves contained (left). It can be decomposed into 9 small copies of itself of the same orientation, it is a lattice tile (right).

There is just one CCW prototile and it appears in just one orientation, so it is a lattice tile. This is shown in Figure 1.4.1-F. The same is true for the CW tile, which (in the special case of folding curves!) is identical.

1.4.2 Dekking's tile criteria

In [9], Dekking gives two theorems about folding curves.

Dekking-1: If the first iterate of both tiles $([L+R+]^2 \text{ and } [L-R-]^2)$ are self-avoiding, then all iterates are self-avoiding. In other words, the folding curve is self-avoiding.



Figure 1.4.2-A: First (left) and third (right) iterates of the tile $[L+R+]^2$ for a folding curve that is self-avoiding but not plane-filling. Note the two squares inside the first iterate that are not traversed. Coloring by curves of the first iterate.

As an example, we use the self-avoiding (but not plane-filling) curves with maps $L \mapsto L-R+L+R-L$ and $R \mapsto R+L-R-L+R$. The first and third iterates of the tile $[L+R+]^2$ are shown in Figure 1.4.2-A.

Dekking-2: If the interiors of both tiles are filled, the curve is plane-filling.



Figure 1.4.2-B: First (left) and third (right) iterates of the tile $[L+R+]^2$ for a folding curve that is self-avoiding and plane-filling. Coloring by curves of the first iterate.

The tile $[L+R+]^2$ for the plane-filling curves with maps $L \mapsto L+R-L-R-L+R+L+R-L$ and $R \mapsto R+L-R-L-R+L+R+L-R$ is shown in Figure 1.4.2-B.

The following generalization of Dekking's tile criteria for curve-sets appears natural.

Dekking-1 CS: If the first iterates of all prototiles of a grid are self-avoiding then all iterates are self-avoiding. In other words, the curve-set is self-avoiding.

The proof Dekking gave can be extended to the more general situation of curve sets [Dekking, 30-November-2023, personal communication].

Note that the condition is equivalent to demanding that the first iterates of all transitions are self-avoiding.



Figure 1.4.2-C: First (left) and third (right) iterate of the tile $[L+R+]^2$ for the curve-set with maps $L \mapsto L+R-L$ and $R \mapsto R$. Coloring by the curves of the first iterate.

The following is *wrong* for curve-sets.

Dekking-2 CS (A): If the interiors of all prototiles are filled, the curve-set is plane-filling.

Dekking [8, Figure 15, p. 14] gave an example where the second criterion does not work. We give a slightly simpler one. Consider the curve with maps $L \mapsto L+R-L$ and $R \mapsto R$. The first and third iterates of the prototile $[L+R+]^2$ are shown in Figure 1.4.2-C (iterates of the other prototile $[L-R-]^2$ give essentially identical images). The iterates of L give arbitrarily long zigzag lines (and L is constant). This is certainly not a plane-filling curve.

One would hope for the following to be true.

Dekking-2 CS (B): If no letter is constant and the interiors of all prototiles are filled, the curve-set is plane-filling.

Sadly, this is *still* wrong. A counterexample is the curve-set (on a coloring of the square grid treated in Section 2.1.5 on page 22) with maps

```
A |--> A+B-A
B |--> B-A+B-A+B-A+B-A+B
C |--> C+D-C+D-C+D-C+D-C
D |--> D-C+D
```



Figure 1.4.2-D: First iterate of the tile $[A+B+C+D+]^1$ of a curve-set without any constant letter that is not plane-filling. Coloring by curves of the first iterate.

As shown in Figure 1.4.2-D, the situation is essentially the same as in the last counterexample: all four curves and the two tiles have the limit shape of a line. This curve-set was found by Daniel Fischer [10].

The situation is not dire, though. Curve-sets that are not plane-filling are recognized easily.

The substitution matrices of both (counter)examples are reducible (as defined in [12,

Definition 6.2.21, p. 403]). For the last example it is

$\begin{bmatrix} 2 \end{bmatrix}$	4	0	0]
1	5	0	0
0	0	5	1
0	0	4	2

This is clearly reducible, it even is of block-diagonal form. For the example with one letter constant the substitution matrix is reducible as well.

It appears that an irreducible substitution matrix implies that a curve-set is planefilling. However, this is not a necessary condition: we'll present plane-filling curve-sets with one or more letters constant; their substitution matrices are reducible.

1.4.3 Creating refined grid colorings

For the sake of simplicity, we use the square grid for our description of a systematic way of finding colorings.

For each coloring there is a minimal vector $\vec{s} = [r, c]$ with r, c > 0 such that every square is moved to a copy of itself by any shift (along diagonals) $[i \cdot r, j \cdot c]$ with $i, j \in \mathbb{Z}$. Figure 1.3.1-A on page 8 shows a coloring with 24 colors where $\vec{s} = [2, 3]$.

A computer search

A systematic search for colorings can proceed as follows. Choose a vector $\vec{s} = [R, C]$ with R, C > 0 and work on the toroidal graph obtained by connecting edges at opposite sides as required. Tentatively assign colors to edges in all ways satisfying the unique transition property. Some of the colorings found are duplicates of another coloring after a suitable permutation of letters; these must be discarded.

The search will find all colorings whose minimal vectors $\vec{s} = [r, c]$ are such that r divides R and c divides C.

Let G be the group of symmetries of the toroidal graph.

One will always find the maximally refined coloring with all edges in different classes, corresponding to the trivial subgroup of G consisting of just the neutral element (see Figure 1.3.1-A on page 8). For this coloring we have $\vec{s} = [R, C]$.

One will also always find the minimal coloring corresponding to the whole group G (for the square grid just one edge class). For this coloring we have $\vec{s} = [1, 1]$.



Figure 1.4.3-A: The two colorings using two colors: $\vec{s} = [2, 2]$ (left) and $\vec{s} = [1, 1]$ (right).

We did such a search for the square grid. There are two colorings using two colors, as shown in Figure 1.4.3-A. The vector \vec{s} is given for each coloring.

Figures 1.4.3-B, 1.4.3-C, and 1.4.3-D show all five colorings with four colors and the one coloring with five colors.



Figure 1.4.3-B: Two colorings using four colors: $\vec{s} = [2, 2]$ (left) and $\vec{s} = [2, 1]$ (right).



Figure 1.4.3-C: Two colorings using four colors: $\vec{s} = [2, 2]$ (left) and $\vec{s} = [1, 1]$ (right).



Figure 1.4.3-D: A coloring using four colors with $\vec{s} = [2, 2]$ (left) and the unique coloring using five colors with $\vec{s} = [5, 5]$ (right).

Ad hoc methods

Grid coloring with many colors tend to lead to curve-sets that look just random, see [4, Figure 15, p. 137] for one example on the square grid using eight colors.

So one may focus on colorings with only a few colors. Some ad hoc constructions tend to work. For example, the coloring of the triangle grid shown in Figure 2.2.4-A on page 32 was obtained by dropping the rotational symmetry around the center of the triangles. To verify a tentative coloring, one just has to check the unique transition property.

Indeed, we obtained all grid colorings in an ad hoc manner, except for the square grid.

2 Examples of curve-sets

A relentless onslaught of examples lies ahead of you. Enjoy.

2.1 The square grid

Here we use turns by 90 degrees $(2\pi/4)$.

2.1.1 The minimal coloring with one color



Figure 2.1.1-A: First iterate of a curve of order 29 on the square grid (left) and third iterate as area drawing (right). Coloring by edges in the first iterate.

Examples of curves on the minimal coloring are given in [2, Section 6.2, pp.60–64]. A curve with 2-fold rotational symmetry is shown in Figure 2.1.1-A. The coloring is by the edges of the motif in the left. The map of the curve is

2.1.2 A coloring with two colors



Figure 2.1.2-A: Coloring of the square grid with two colors (left) and its three prototiles (right).

We use the coloring shown in Figure 2.1.2-A (this is the coloring shown on the left in Figure 1.4.3-A on page 15). Our curve-set of order 25 has the following maps:

 $\mathsf{G} \hspace{0.1in} | \hspace{0.5em} - \hspace{-0.5em} > \hspace{0.1in} \mathsf{G} - \mathsf{F} + \mathsf{F} - \mathsf{G} + \mathsf{G} + \mathsf{G} - \mathsf{F} + \mathsf{F} + \mathsf{F} - \mathsf{G} - \mathsf{F} - \mathsf{G} + \mathsf{G} + \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf{F} + \mathsf{F} - \mathsf{G} - \mathsf{F} + \mathsf{F} - \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf{F} + \mathsf{F} - \mathsf{G} - \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf{G} - \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf{G} - \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf{G} - \mathsf{G} - \mathsf{G} - \mathsf{F} - \mathsf{G} - \mathsf$

The first iterates of the two curves are shown in Figure 2.1.2-B. These curves are mutually self-similar, see Figure 2.1.2-C.

The CW prototile $[F-G-]^2$ is self-similar, it can be decomposed into smaller copies of itself, appearing in two orientations, as shown in Figure 2.1.2-D.

The CCW prototiles are $[F+]^4$ and $[G+]^4$. Their iterates will have 4-fold rotational symmetries and be mutually self-similar. However, for this curve-set the iterates of the two CCW prototiles are squares, so the mutual self-similarity is boring (the



Figure 2.1.2-B: Motifs for the two curves for F (left) and G (right).



Figure 2.1.2-C: Mutual self-similarity of the curves F (left) and G (right). Coloring cycling through 12 colors in the small copies.



Figure 2.1.2-D: The CW prototile $[F-G-]^2$ is self-similar and tiles the plane.

 5×5 chessboard pattern). For an example of non-trivial mutual self-similarity see Figure 2.1.2-E. The curve-set is of order 25 and has maps

Curve sets where both curves have reflection symmetry exist, an example with order 49 is shown in Figures 2.1.2-F and 2.1.2-G. The maps are



Figure 2.1.2-E: Mutual self-similarity of the CCW prototiles of a curve-set. Small copies of the left and right prototile respectively in blue and red.



Figure 2.1.2-F: First iterate of the two curves of a curve-set of order 49. Curves F (left) and G (right). Coloring by orientation of the edges.



Figure 2.1.2-G: Fourth iterate of the two curves of a curve-set of order 49. Curves F (left) and G (right). Coloring by orientation of the small curves.

2.1.3 The other coloring with two colors

This is already treated in Section 1.4.1 on page 10, here we just give a numeration system corresponding to the tile of a folding curve.

Complex numeration system

The numeration system for the folding curve with maps

- L |--> L+R+L-R+L+R-L+R-L
- $R \quad | --> \quad R+L-R+L-R-L+R-L-R$

whose motifs are shown in Figure 1.4.1-D on page 11.



Figure 2.1.3-A: First (left) and fifth (middle) iterate of the CCW tile and the fundamental region of the corresponding numeration system (right).

The first iterate of the tile is shown on the left in Figure 2.1.3-A. We take the central square as the origin of the complex plane. The coordinates of all 9 squares give the set

$$D = \{0, +1 - i, -1 + i, +2i, -2i, +1 + 3i, -1 - 3i, +2 + 2i, -2 - 2i\}$$

A numeration system with radix +3 and digits D has the fundamental region shown on the right. The *fundamental region* of a numeration system is the set of numbers whose expansion has zero integral part in their expansion (in that numeration system), see [2, Section 3.3, pp. 20-26].

A complex numeration system corresponding to the tile of a folding curve of order 5 is shown in [3, Figure 7, p.182].

2.1.4 A coloring with four colors



Figure 2.1.4-A: A coloring of the square grid with four colors (left). The prototiles are $[A+B+C+D+]^4$ and $[A-B-C-D-]^4$ (right).

fundamental region

With the coloring shown in Figure 2.1.4-A (same as left in Figure 1.4.3-D on page 16) the sequence of letters in any curve is always A, B, C, D, A, B, C, D, ..., regardless of the turns. Each prototile appears in two orientations.

A curve-set of order 49

Our curve-set of order 49 has the maps

- A |--> A+B-C+D+A-B-C-D+A+B-C+D-A-B-C+D-A+B-C+D-A+B+C-D-A+B+C+D-A-B+ \ C-D+A+B+C-D+A-B+C-D-A+B-C-D-A+B+C+D-A+B-C-D-A+B+C+D-A B |--> B+C-D+A+B-C-D-A+B-C-D+A-B+C+D-A-B+C
- D+A-B+C+D-A-B-C+D-A-B+C-D+A+B+C-D+A-B
- $D | -> D+A-B+C+D-A-B-C+D+A-B-C-D+A-B+C-D+A+B+C-D+A+B+C-D-A+B+C+D-A+ \\ B-C-D+A-B-C-D+A+B+C-D+A-B-C-D+A+B+C-D$

The mutual self-similarity of the four curves is shown in Figure 2.1.4-B. All curves have reflection symmetry.



Figure 2.1.4-B: Mutual self-similarity of the curves (from left to right) A (small copies black), B (small copies blue), C (small copies red), and D (small copies green). First iterates in top row, third iterates in bottom row.

A curve-set of order 5 and a numeration system

Our curve-set has the maps

```
A |--> A-B+C
B |--> D+A-B-C-D+A-B+C+D+A-B
C |--> C-D+A
D |--> B-C+D
```

Their motifs can be gleaned from the left depiction in Figure 2.1.4-C. A lattice tile is created by combining two CW tiles $[A-B-C-D-]^1$ into the axiom A-B-C-D+A-B-C-D. The fifth iterate of this is shown in the middle of the figure. The corresponding numeration system is shown on the right. It has radix -2 + i and the digit set is

$$D = \{0, +1, -1, +1 - i, -1 + i\}$$

The digits correspond to the positions of the centers of the small copies of lattice tiles.



Figure 2.1.4-C: First iterate of the lattice tile with axiom A-B-C-D+A-B-C-D, coloring by curves (left). Sixth iterate of the lattice tile, decomposed into smaller copies of itself (middle) and the corresponding numeration system (right).



Figure 2.1.4-D: Attempt to decompose the first iterate of the lattice tile A+B+C+D-A+B+C+D- into smaller copies of itself (left) and fourth iterate with this coloring (middle). Several such tiles, loosely assembled to tile the plane (right).

We remark that the CCW lattice tile cannot be decomposed into smaller copies of itself. The left in Figure 2.1.4-D shows an attempt to do so, starting from the two central squares. Both the light green and the dark green parts are disconnected. It is still a lattice tile, the whole place can be covered with translated copies of it. Unsurprisingly, we cannot give a numeration system corresponding to it.

2.1.5 Another coloring with four colors



Figure 2.1.5-A: A coloring of the square grid using four colors (left) and its prototiles $[A+B+C+D+]^1$ (middle) and $[A-D-C-B-]^1$ (right).

The coloring together with its prototiles is shown in Figure 2.1.5-A, it is the same

as shown in Figure 1.4.3-C on page 16 (right). Either prototile occurs in just one orientation, both are lattice tiles.

All curve-sets presented here where found by Daniel Fischer [10].

A curve-set of order 4



Figure 2.1.5-B: A curve-set drawn with the axiom A+B+C+D-C-B-A connecting both prototiles (left). The seventh iterate of this is shown on the right.

We consider the curve-set of order 4 with the following maps

A |--> D+A-D-C-B+C+D B |--> A-D+A+B-A C |--> B+C-B D |--> C

An image combining both prototiles is shown on the right in Figure 2.1.5-B on page 23. The shape of the CW prototile is the twindragon, that of the CCW prototile a square.

A curve-set of order 7



Figure 2.1.5-C: Iterate 6 of the CCW prototile of a curve-set of order 7 (left, coloring by curves). Iterate 6 of the CCW tile of a curve of order 7 on the triangle grid (right, coloring by orientation of the curves).

A curve-set of order 7 with maps

A |--> A-D+A B |--> B-A+B

```
C |--> C-B-A+B+C-B+C+D+A-D-C-B+C
D |--> D+A-D+A-D-C-B+C+D
```

The CCW prototile shown on the left in Figure 2.1.5-C looks like a deformed Gosper island, a figure that is not expected on the square grid. The Gosper island on the right is the CCW tile $[F+]^3$ on the triangle grid (minimal coloring) using the curve with map

F |--> F+F-F-F+F+F-F

The next example is of the same nature.

A curve-set of order 3



Figure 2.1.5-D: Iterate 10 of the CCW prototile for a curve of order 3 (left). Iterate 10 of the CCW prototile for terdragon (right).

A curve-set of order 3 with maps

```
A |--> A+B-A
B |--> B+C-B
C |--> C+D+A-D-C
D |--> D
```

The CCW prototile is shown on the left in Figure 2.1.5-D, only three curves are apparent because D is a constant. All three curves look like distorted terdragons: The right image shows the CCW tile $[F+]^3$ of the terdragon (again on the triangle grid). Its map is

F |--> F+F-F

A curve-set of order 6

Both prototiles of a curve-set of order 6 are shown in Figure 2.1.5-E. The maps of this curve-set are

```
A |--> A-D-C+D-C-B+C+D+A-D+A
B |--> B-A+B-A-D+A+B
C |--> C-B+C-B+C
D |--> D
```

There is no valid excuse for presenting this, but we do like the looks.



Figure 2.1.5-E: Iterate 6 of the CCW (left) and CW prototile (right) of a curve-set of order 6.

2.2 The triangle grid

Here we use turns by 120 degrees $(2 \pi/3)$.

2.2.1 The minimal coloring with one color

Examples of curves on the minimal coloring are given in [2, Section 6.1, pp.56–59].



Figure 2.2.1-A: Third iterates of the tiles $[F+]^3$ (left) and $[F-]^3$ (right) of a curve of order 25. Both have reflection symmetry. Coloring by orientation of curves.

We give an example where both tiles have reflection symmetry. The curve of order 25 with map

F |--> FOF+F+F-F-FOF-FOF+F+F-F-FOF-FOF+F+F-FOFOF+F-F+FOF gives the tiles shown in Figure 2.2.1-A.

2.2.2 A coloring with three colors



Figure 2.2.2-A: A coloring of the triangle grid with three colors (left) and its four prototiles $[F+]^3$, $[G+]^3$, $[H+]^3$, and $[F-H-G-]^1$ (right).

We use the grid coloring shown in Figure 2.2.2-A, together with its prototiles.

Our curve-set is of order 25 and has maps

```
F |--> F+F-H+H-G-FOG+GOHOF
```

```
G |--> G-F+F-H+H+H-G+G+G-FOG-FOG-F-H+H+HOFOG
```

H |--> H-G+G-F+F-H+H+H0F0G-F0G+G0H-G-F+F-H+H

The motifs of the curves are shown in Figure 2.2.2-B.



Figure 2.2.2-B: Motifs for the curves of a curve-set of order 25. From left to right, motifs for curve F, G, and H.

The three CCW prototiles, $[F+]^3$, $[G+]^3$, and $[H+]^3$, have 3-fold rotational symmetry and are mutually self-similar as shown in Figure 2.2.2-C.

The arrangement (F+F+F)-(H+H+H)-(G+G+G) of the CCW prototiles tiles the plane by itself (it is a lattice tile), it is shown in the lower right

The one CW prototile $[F-H-G-]^1$ tiles the plane, appearing in three orientations. Its selfsimilarity is shown in Figure 2.2.2-D (left). The arrangement (F-H-G)+(G-F-H)+(H-G-F)of the prototile $[F-H-G-]^1$ in its three orientations tiles the plane, appearing in just one orientation (so it is a lattice tile again), see Figure 2.2.2-D (right). Neither the prototile nor the arrangement have any rotational symmetry.



Figure 2.2.2-C: Mutual self-similarity of the CCW prototiles $[F+]^3$ (upper left), $[G+]^3$ (upper right), and $[H+]^3$ (lower left). Each is composed of smaller copies of all three of the prototiles. The arrangement of three prototiles (lower right) is a lattice tile.



Figure 2.2.2-D: Self-similarity of the CW prototile $[F-H-G-]^1$ (left). The prototile tiles the plane, appearing in three orientations. The arrangement of three prototiles (right) is a lattice tile.



2.2.3 Another coloring with three colors

Figure 2.2.3-A: Coloring of the triangle grid (left), its two prototiles $[A+B+C+]^1$ and $[A-B-C-]^1$ (right). Both prototiles appear in three orientations.

The grid coloring is shown in Figure 2.2.2-A. With this coloring the sequence of letters in any curve is always A, B, C, A, B, C, \ldots , regardless of the turns.

A curve-set of order 16



Figure 2.2.3-B: Motifs of the curves A (left), B (middle), and C (right).

Our curve-set is of order 16 and has maps

```
A |--> A+B-C+A-B-COA+BOC-A-B+C-A+B+COAOB+C-A
```

- B |--> B+C-A+B-C-A0B+C-A+B+C0A-B-C+A-B0C+A+B-C+A0B-C-A+B
- C |--> COAOBOC

The first iterates are shown in Figure 2.2.3-B.

Each prototile appears in three orientations, the iterates of either tile the plane. The CCW prototile $[A+B+C+]^1$ is shown in Figure 2.2.3-C, the CW prototile $[A-B-C-]^1$ in Figure 2.2.3-D.

Here one can easily build six lattice tiles, see Figures 2.2.3-E and 2.2.3-F. Each has a 3-fold rotational symmetry.



Figure 2.2.3-C: The CCW prototile $[A+B+C+]^1$ split into curves (left) and its decomposition into smaller copies of itself showing its self-similarity (right).



Figure 2.2.3-D: The CW prototile $[A-B-C-]^1$ split into curves (left) and its decomposition showing its self-similarity (right).



Figure 2.2.3-E: Left to right: the lattice tiles $(A+B+C-)^3$, $(B+C+A-)^3$, $(C+A+B-)^3$, $(A-B-C+)^3$, $(B-C-A+)^3$, and $(C-A-B+)^3$. The bottom row shows the corresponding area drawings.



Figure 2.2.3-F: Third iterates of the lattice tiles. Top row, left to right: $(A+B+C-)^3$, $(B+C+A-)^3$, and $(C+A+B-)^3$. Bottom row, left to right: $(A-B-C+)^3$, $(B-C-A+)^3$, and $(C-A-B+)^3$.

Another curve-set of order 16

It is possible for all curves of a curve-set to have a reflection symmetry. The curves of such a curve-set of order 16 having maps

- A |--> AOB+C-A-BOC+A+B-C+AOB-C-A+B-COAOB+C+A-B+C-A
- B |--> BOCOA+BOC-A-B+C-AOBOC+A+B-C+A-B
- C |--> COAOB+C-A-BOC+A+B-C

is shown in Figure 2.2.3-G.

The covering of the plane with second iterates of the curves is shown in Figure 2.2.3-H.



Figure 2.2.3-G: Mutual self-similarity of a curve-set where all curves have reflection symmetry. First (left) and fourth (right) iterates.



Figure 2.2.3-H: Second iterates of the curves on the grid.

2.2.4 Yet another coloring with three colors



Figure 2.2.4-A: A coloring of the triangle grid (left) and its two prototiles $[A+B+C+]^1$ and $[A-C-B-]^1$ (right).

The coloring of the grid shown in Figure 2.2.4-A has just one CW and CCW prototile. Note that either prototile appears in just one orientation, so both are lattice tiles.

Curve-set of order 7 and a numeration system

The first curve-set is of order 7 and has the maps

- A |--> A+B-A+B+C-B+C-B-A+B-A
- B |--> B+COCOC-B
- C |--> C+AOAOA-C

The first iterate of CCW tile is shown in Figure 2.2.4-B.



Figure 2.2.4-B: First iterate of the prototile $[A+B+C+]^1$, colored by curves (left). The same, colored to separate triangles (right).

Now take the lower red triangle of the right image as origin of the complex plane. The coordinates of all 7 triangles give the set

$$D = \{0, +\omega_6, -\omega_6, +\omega_3, -\omega_3, +2\omega_3, 1+\omega_6\}$$

where $\omega_k = \exp(2\pi i/k)$ (a primitive complex kth root of unity). We use D as digits and the radix $-1 + 3\omega_6$ for the numeration system.



Figure 2.2.4-C: Self-similarity of the prototile $[A+B+C+]^1$ (left) and fundamental region of the corresponding numeration system (right).

The fifth iterate of the CCW tile is shown on the left in Figure 2.2.4-C, together with the fundamental region of the numeration system on the right.

A curve-set of order 36 whose curves have reflection symmetry

Our second curve-set of order 36 was constructed so that all curves have reflection symmetry. Its maps are

```
A |--> A+B-A+B-A+B-A+B-A-COCOCOCCC+A-C+A-C+A-C+A+B0B0B0B-A0A0A0A0A
```

```
B |--> B+C-B+C-B+C-B+C-B-A0A0A0A0A+B-A+B-A+B-A+B+C0C0C0C-B0B0B0B0B
```

```
C |--> C+A-C+A-C+A-C+A-C-B+C-B+C-B-A+B-A-C+A+B0B0B0B+C-B+C-B+C-B+C+ \
A-C+A-C+A+B-A+B+C-B-A0A0A-C0C0C0C0C
```

The mutual self-similarity of the three curves is shown in Figure 2.2.4-D.

The self-similarity of the CCW tile $[A+B+C+]^1$ is shown in Figure 2.2.4-E. The tile is a lattice tile and corresponds to the fundamental region of a numeration system with 36 digits.



Figure 2.2.4-D: Mutual self-similarity of the curves $\tt A$ (top left), $\tt B$ (top right), and $\tt C$ (bottom).



Figure 2.2.4-E: The CCW tile $[A+B+C+]^1$: first iterate, decomposed into curves (left) and its self-similarity (right).

A curve-set with uneven growth

Our curve-set has order 6 and maps

```
A |--> AOA+B-A
B |--> B-A+B+C-B
C |--> C+A-COC-B-A+B+COC
```

Figure 2.2.4-F shows the first, second, and third iterate of the prototile $[\tt A+B+C+]^1.$ So far, nothing suspect.



Figure 2.2.4-F: First (left), second (middle), and third (right) iterate of the CCW prototile.

-



However, at higher iterates an uneven growth shows, see Figure 2.2.4-G. The curve-set *is* plane-filling, but not in a nice way: the growth in one direction is much faster than in the other. This curve-set was found by Daniel Fischer [10].

Using the same motif on different colorings



Figure 2.2.4-H: First and second iterates of the CCW tiles of the same shape but with transitions for different grid colorings. Note the different sequences of letters.

Using the same (shape of) motif with different colorings leads to different shapes of curves (and tiles) in general. Figure 2.2.4-H shows the CCW tiles $[A+B+C]^1$ with lettering for the coloring in this section (first and second iterate, left) and lettering according to the coloring of Section 2.2.3 on page 28 (again first and second iterate, right). The coloring used in the images is by curves: blue for A, red for B, and green for C.

The seventh iterates of the CCW tile for the coloring in this section is shown on the left in Figure 2.2.4-I (the CW tile looks identical). The green part corresponding to the 1-dimensional curve for C isn't visible at this iterate. Middle and right respectively show

the CCW and CW tiles for the coloring of Section 2.2.3. The green part corresponding to the curve for C suggests it is 2-dimensional.



Figure 2.2.4-I: The CCW tile using the grid coloring in this section (left). The CCW (middle) and CW (right) tile using the grid coloring from Section 2.2.3.

The lozenge-shaped tile may not look terribly exciting but there is a numeration system with the fundamental region of just that shape with radix -2 and the set of digits $\{0, +1, +\omega_6, +1+\omega_6\}$ where $\omega_6 = \exp(2\pi i/6)$. In addition the shapes of both curves **A** and **B** are triangles, which does not happen for any curve on the minimal coloring.

2.2.5 A coloring with four colors



Figure 2.2.5-A: A coloring of the triangle grid with four colors (left), its CCW tiles $[A+]^3$ and $[B-C-D-]^1$ (middle), and its CW tiles $[B-]^3$ and $[A-C-D-]^1$ (right).

The coloring we use is shown in Figure 2.2.5-A, together with its prototiles.

Our curve-set of order 9 has maps

- A |--> A+A-C-DOC+D+B-BOA
- B |--> B+C-D-AOB+C+D-AOB
- C |--> C+D-A-COD+B+C-D+B-B-BOA+A+A-C
- D |--> DOCOD

The first iterates of the curves are shown in Figure 2.2.5-B.

Though the motifs of all curves have reflection symmetry, only curves C and D keep it with higher iterates. This can be seen in Figure 2.2.5-C where mutual self-similarity of the curves is also shown.

The curves tile the plane as shown in Figure 2.2.5-D. Note that the blue regions each consist of three curves A and the red regions each consist of three curves B.


Figure 2.2.5-B: From left to right: motifs of the curves A, B, C, and D.



Figure 2.2.5-C: From left to right: Third iterates of the curves A (small copies blue), B (small copies red), C (small copies green), and D (small copies pale pink).



Figure 2.2.5-D: Tiling of the plane with our curves.

2.3 The trihexagonal grid

Here turns are by 60 degrees $(2\pi/6)$.

2.3.1 The minimal coloring with one color

Examples of curves on the minimal coloring of the trihexagonal (or trihex- or (3.6.3.6)grid or kagome lattice, see Figure 2.3.1-A) are given in [2, Section 6.3, pp. 66–71]. Figure 2.3.1-B shows the motif (left) and both tiles of a curve of order 67 with the map

We chose this curve because the motif does not look promising, still, this is a planefilling curve. The second iterates of both tiles are shown in Figure 2.3.1-C.



Figure 2.3.1-A: Minimal coloring of the (3.6.3.6)-grid (left), the CCW prototile $[F+]^6$ (middle), and the CW prototile $[F--]^3$ (right).



Figure 2.3.1-B: Motif of a curve of order 67 (left), the CCW tile $[F+]^6$ (middle), and the CW tile $[F--]^3$.



Figure 2.3.1-C: Second iterates of the CCW (left) and CW (right) tile of the curve.

2.3.2 A coloring with two colors

The coloring is shown in Figure 2.3.2-A, together with its prototiles $[A+B+]^6$ (CCW, hexagon), $[A--]^3$ and $[B--]^3$ (CW, triangles). The CCW tile appears in just one orientation, so it is a lattice tile.



Figure 2.3.2-A: Coloring of the (3.6.3.6)-grid using two colors (left). The CCW prototile (middle) and the two CW prototiles (right).

A curve-set where the CW prototiles have maximal symmetry

Our curve-set is of order 25 and has maps

Figure 2.3.2-B: From left to right: motif for curve A, motif for tile $[A--]^3$, motif for curve B, and motif for tile $[B--]^3$.



Figure 2.3.2-C: Third iterates of the CW tiles $[A--]^3$ (left) and $[B--]^3$ (right). Coloring by orientation of curves.

The motifs for curves A and B and CW tiles $[A--]^3$ and $[B--]^3$ are shown in Figure 2.3.2-B. The third iterates of the CW tiles are shown in Figure 2.3.2-C. The symmetry of the CW tiles is maximal, including reflection symmetry (this is only possible if the order is a square). The mutual self-similarity of the CW tiles is shown in Figure 2.3.2-D. The CCW tile $[A+B+]^3$ has 3-fold symmetry, as shown in Figure 2.3.2-E.



Figure 2.3.2-D: Decompositions of the CW tiles $[A--]^3$ (left) and $[B--]^3$ (right) into smaller copies of themselves. Coloring by orientations of the curves in the small copies.



Figure 2.3.2-E: Third iterate of the CCW tile $[A+B+]^3$, colored by orientations of the curves contained (left). The same, decomposed into 25 smaller copies of itself (right).

A curve-set with one letter constant



Figure 2.3.2-F: First (left), second (middle), and third (right) iterate of a curve-set with one letter constant. Coloring by letters, blue for A and red for the constant B.

Our curve-set is of order 9 and has the maps

The first, second, and third iterates of the curve A are shown in Figure 2.3.2-F. Note how the red areas vanish with higher iterates. The shape of the curve has a 2-fold symmetry in the limit.



Figure 2.3.2-G: Fourth iterate of the curve A, decomposed into 16 smaller copies of itself. Coloring by orientation of the small copies.

The fourth iterate (Figure 2.3.2-G) is indistinguishable from the curve of order 16 on the triangle grid with map

All nonzero turns are by ± 120 , so we can replace all turns by a single symbol and use turns by 120 degrees. Finally, replace A by F to arrive at the map given above.

Removing all edges B in the grid (2.3.2-A on page 39) gives the triangle grid with minimal coloring, so this is not too surprising.

2.3.3 A coloring with three colors



Figure 2.3.3-A: A coloring of the trihex grid with three colors (left) and its prototiles $[F+G+H+]^2$ (middle) and $[F--G--H--]^1$ (right).

We use the coloring shown in Figure 2.3.3-A. The CCW prototile $[F+G+H+]^2$ now has 2-fold symmetry and the CW prototile $[F-G-H--]^1$ appears in two orientations.

With this coloring the sequence of letters in any curve is always $F,\,G,\,H,\,F,\,G,\,H,\,\ldots,$ regardless of the turns.

We use a curve-set of order 25 with maps

```
F |--> F+G+H--F+G+H--F--G+H--F+G+H+F+G--H--F+G+H+F--G+H+F+G+H+F--G+H--F
```

- $\texttt{G} \quad | \texttt{-->} \quad \texttt{G} + \texttt{H} + \texttt{F} + \texttt{G} \texttt{H} \texttt{F} + \texttt{G} + \texttt{H} + \texttt{F} \texttt{G} \texttt{H} + \texttt{F} + \texttt{G} + \texttt{H} + \texttt{H} + \texttt{G} + \texttt{H} + \texttt{G} + \texttt{H} + \texttt{H$
- $\label{eq:hardenergy} H \hspace{0.1in} | \hspace{0.1in} \hspace{0.1in} > \hspace{0.1in} H + F + G H + F + G H F + G + H F + G H + F + G + H + F + G H$



Figure 2.3.3-B: Motifs for the curves F (left), G (middle), and H (right).

The motifs of the curves are shown in Figure 2.3.3-B.



Figure 2.3.3-C: First iterate (left, coloring by curves) and self-similarity (right, coloring of the small tiles by orientation) of the CW tile $[F--G--H--]^1$.

The self-similarity of the CW prototile is shown in Figure 2.3.3-C, note that the red and blue small copies appear in two orientations.



Figure 2.3.3-D: Third iterate of the CCW tile $[F+G+H+]^2$, colored by curves contained (left). The same, decomposed into 25 smaller copies of itself (right).

Figure 2.3.3-D shows the 2-symmetric CCW prototile. It is a lattice tile.

2.3.4 A coloring with six colors

In the last few sections we have reduced the rotational symmetry of the CCW tile from 6-fold, to 3-fold and 2-fold. A coloring with six colors is shown in Figure 2.3.4-A, it guarantees no rotational symmetry for the CCW tile. It still is a lattice tile.



Figure 2.3.4-A: Coloring of the (3.6.3.6)-grid with six colors (left), its CCW prototile $[A+B+C+D+E+F+]^1$ (middle), and its CW prototiles $[A-E-C--]^1$ and $[B-F-D--]^1$.



Figure 2.3.4-B: First (left) and fourth (middle) iterate of the CCW tile $[A+B+C+D+E+F+]^1$, colored by curves contained. The self-similarity of the tile is shown on the right: it can be decomposed into 16 small copies of itself.

Our curve-set is of order 16 and has maps

```
A |--> A--E+F+A+B+C+D--B+C--A+B+C--A--E+F--D+E+F+A+B+C--A--E+F+A
```

```
B |--> B+C+D--B--F+A+B
```

```
C |--> C--A+B+C+D+E+F--D+E--C+D--B--F+A+B+C+D+E--C--A+B+C
```

```
\mathsf{D} \hspace{0.1in} | \hspace{0.1in} - \hspace{-0.1in} > \hspace{0.1in} \mathsf{D} \hspace{-0.1in} - \hspace{-0.1in} \mathsf{B} \hspace{-0.1in} + \hspace{-0.1in} \mathsf{C} \hspace{-0.1in} + \hspace{-0.1in} \mathsf{D} \hspace{-0.1in} + \hspace{-0.1in} \mathsf{E} \hspace{-0.1in} + \hspace{-0.1in} \mathsf{F} \hspace{-0.1in} - \hspace{-0.1in} \mathsf{B} \hspace{-0.1in} + \hspace{-0.1in} \mathsf{C} \hspace{-0.1in} + \hspace{-0.1in} \mathsf{D} \hspace{-0.1in}
```

```
E |--> E--C+D+E+F+A+B--F+A--E+F--D--B+C+D+E+F+A--E--C+D+E
```

```
F |--> F+A+B--F--D+E+F
```

The motifs of the curves, assembled into the CCW tile, are shown in Figure 2.3.4-B.

The CW tiles are shown in Figure 2.3.4-C. There are two ways of pairing them into lattice tiles. The first way is to connect them so that the turn B+C is contained. This arrangement and its self-similarity is shown in Figure 2.3.4-D.

The second way to connect the tiles contains the turn A+B. This arrangement is shown in Figure 2.3.4-E. It is not self-similar: note that the small copy at the top of the right image in the figure is incomplete, it is missing the red/yellow/pink part which lies at the lower right of the image.



Figure 2.3.4-C: From left to right: first and third iterates of the CW tile $[A--E--C--]^1$, first and third iterates of the CW tile $[B--F--D--]^1$.



Figure 2.3.4-D: Arrangement (axiom) for a lattice tile (left), its third iterate (middle), and its decomposition into 16 smaller copies of itself (right).



Figure 2.3.4-E: Arrangement (axiom) for a lattice tile (left), its third iterate (middle), and a failed attempt to decompose it into smaller copies of itself (right).

2.4 The (3.4.6.4)-grid

2.4 The (3.4.6.4)-grid

Here we use turns by 30 degrees $(2\pi/12)$.

2.4.1 The minimal coloring with two colors



Figure 2.4.1-A: The minimal coloring of the (3.4.6.4)-grid (left). Its CCW prototiles $[A^{++}]^6$ and $[B^{++++}]^3$ (middle), and the CW prototile $[A^{---}B^{---}]^2$ (right).

The minimal coloring of the (3.4.6.4)-grid and its prototiles are shown in Figure 2.4.1-A (same as Figure 1.3.0-A on page 6).

A tedious construction for edge-covering curves on this grid is given in [2, Section 4.2.1, pp.42-43]. Using the proper amount of edge classes (two, not one!) makes life considerably easier.

6-fold symmetry is possible for the CCW tile [B++++]³



Figure 2.4.1-B: From left to right: motifs for the letter A and B, and first iterates for prototiles $[A++]^6$, $[B++++]^3$, and $[A---B---]^2$ of a curve on the (3.4.6.4)-grid.

The rotational symmetries of the prototiles are six-fold for the $[A++]^6$, at least three-fold for $[B++++]^3$, and two-fold for $[A---B---]^2$. The prototile $[B++++]^3$ may have six-fold symmetry. An example is the curve-set of order 9 with maps

A |--> A++A++A++A++A---B---A---B++++B---A

The motifs of the curves and tiles are shown in Figure 2.4.1-B.

The first iterate of $[B^{++++}]^3$ has six-fold symmetry, see Figure 2.4.1-C. Both prototiles happen to have the shape of the Koch snowflake, see [6, Figure 3]. Note that this shape also has reflection symmetry.

The tile $[A--B---]^2$ shown in Figure 2.4.1-E can be decomposed into 9 smaller copies of itself, appearing in three orientations (image on the right).



Figure 2.4.1-C: Fourth iterates of the prototiles $[A++]^6$ (left) and $[B++++]^3$ (right). Coloring by curves and their orientations.



Figure 2.4.1-D: Mutual self-similarity of the prototiles $[A++]^6$ (left, small copies blue) and $[B++++]^3$ (right, small copies red).



Figure 2.4.1-E: Fourth iterate of the prototile $[A---B---]^2$ (left, coloring by orientation of curves contained) and its self-similarity (right, coloring by orientation of small copies of itself).



Figure 2.4.1-F: Mutual self-similarity of the prototiles $[A]^6$ (left, small copies blue) and $[B]^3$ (right, small copies red) of a curve-set of order 39.

The two CCW tiles shown in Figure 2.4.1-F also have 6-fold symmetry, they belong to a curve-set of order 39. Here the shapes of the two tiles are different from each other. The maps of this curve-set, omitting the turns, are

Note that the map for B is a square, as indicated by the comma in its middle. Here we can omit turns, as the transitions are A++A, B++++B, A---B, and B---A and the two letters of each transition uniquely determine the turn (this is of course *not* true in general).

A curve-set of order 37

The following curve-set is presented because the tiles look quite curious. It has order 37 and the maps (omitting turns again)

```
A |--> AAAABAAABBABAAABBAA
```



Figure 2.4.1-G: Third iterates of the tiles $[A++]^6$ (left) and $[B++++]^3$ (right). Coloring by orientation of curves contained.



Figure 2.4.1-H: Decomposition of the prototile $[A++]^6$ into smaller copies of $[A++]^6$ and $[B++++]^3$. Coloring by orientation of curves (left) and by tiles (right, small copies of $[A++]^6$ blue, those of $[B++++]^3$ red).



Figure 2.4.1-I: Decomposition of the prototile $[B^{++++}]^3$ into smaller copies of $[A^{++}]^6$ and $[B^{++++}]^3$. Coloring by orientation of curves (left) and by tiles (right, small copies of $[A^{+++}]^6$ blue, those of $[B^{++++}]^3$ red).

The CCW tiles $[A++]^6$ and $[B++++]^3$ are shown in Figure 2.4.1-G. Their mutual selfsimilarity can be observed in Figures 2.4.1-H and 2.4.1-I. Note that while the prototiles $[A++]^6$ are separated, the prototiles $[B++++]^3$ are not, making the images slightly difficult to interpret. The gray borders drawn between the curves are to mitigate this.

A curve-set with letter A constant

Curve-sets whose L-system leaves some of the letters constant exist. Here we have just one non-constant letter and the curve for it shows simple self-similarity.



Figure 2.4.1-J: First (left) and second (right) iterate of the curve B of a curve-set with A constant. Coloring by orientation of the edges of the first iterate.



Figure 2.4.1-K: Self-similarity of curve B, coloring by orientation of the smaller curves. The prototile $[B++++]^3$ is the apparent triangle in the center.

Figure 2.4.1-J shows the motif and second iterate of curve ${\tt B}$ of a curve-set of order 9 with maps

```
A |--> A
```

B |--> BAAAABABBBABBABB

Figure 2.4.1-K shows the self-similarity of the curve. Note that the curve has the shape of half of the curve shown on the right in Figure 2.4.1-E on page 46.

A curve-set with letter ${\tt B}$ constant



Figure 2.4.1-L: First (left) and second (right) iterate of the curve A of a curve-set with B constant. Coloring by orientation of the edges of the first iterate.

Figure 2.4.1-L shows the motif and second iterate of curve ${\tt A}$ of a curve-set of order 19 with maps

A |--> AABBAAABABBABBBAABABBBBABBAAAAAABBA B |--> B



Figure 2.4.1-M: Decomposition of the curve A into 19 smaller copies of itself, showing its self-similarity. Coloring by orientation of the small copies.

The self-similarity of the curve is shown in Figure 2.4.1-M. The apparent lozenges (for example, at the top right) are the tiles $[A---B---]^2$, colored by their orientation. The tile $[A++]^6$ (6-fold symmetry) can be seen at the lower left.

2.4.2 A coloring with four colors



Figure 2.4.2-A: Coloring of the (3.4.6.4)-grid with four colors (left) and the CCW (middle) and CW (right) prototiles.

The coloring shown in Figure 2.4.2-A splits the triangular CCW prototile of the minimal coloring into two classes, prototiles $[B++++]^3$ and $[b++++]^3$.

First curve-set

The curve-set of order 9 has maps

- A |--> A---B++++B++++B---a++A---B---a---b++++b---A++a++A
- a |--> a++A++a---b---A---B++++B---a---b++++b++++b---A++a
- B |--> B++++B---a++A---B
- b |--> b++++b---A++a---b



Figure 2.4.2-B: Motifs, from left to right, for A, a, B, and b.



Figure 2.4.2-C: Mutual self-similarity of the CCW prototiles $[A-a-]^3$ (left), $[B----]^3$ (upper right), and $[b----]^3$ (lower right). Coloring by curves and their orientation.



Figure 2.4.2-D: A version of the previous figure that may be easier to decipher: coloring by tiles.

The CCW tiles shown in Figures 2.4.2-C and 2.4.2-D have reflection symmetry. Moreover the tile $[\tt A++a++]^3$ has 6-fold rotational symmetry.

Second curve-set

The curve-set of order 9 has maps

```
A |--> A---B++++B+++B---a++A---B---a---b++++b---A++a++A
a |--> a++A++a---b---A---B++++B---a++A++a
B |--> B++++B---a---b++++b+++b---A---B
```

```
b |--> b++++b---A++a---b
```

This curve-set was created by modifying the interior of the CW prototile as shown in



Figure 2.4.2-E: Motifs, from left to right, for A, a, B, and b.



Figure 2.4.2-F: CW tile $[A---B---a---b---]^1$ from the curve-set in the previous section (left). The same with a triangle removed from curve **a** (middle). The triangle attached to the curve **B** (right), this is the CW tile for our current curve-set.

Figure 2.4.2-F. This technique of removing a polygon from one curve and reattaching it to another is quite useful for modifying curve-sets.



Figure 2.4.2-G: Mutual self-similarity of the CCW prototiles $[A++a++]^3$ (left), $[B++++]^3$ (upper right), and $[b++++]^3$ (lower right). Coloring by curves and their orientation.

The CCW tiles again have reflection symmetry, they are shown in Figures 2.4.2-G



Figure 2.4.2-H: A version of the previous figure that may be easier to decipher. Coloring by tiles.



Figure 2.4.2-I: Self-similarity of the CW prototile $[A---B---a--b---]^1$, the small copies appear in three orientations. Coloring by orientation.

and 2.4.2-H.

The CW prototile is shown in Figure 2.4.2-I. It has the same shape as the CW prototile for the previous section (we did the change inside it).

2.4.3 A coloring with six colors



Figure 2.4.3-A: Coloring using six colors (left). The three CW prototiles have 2-fold rotational symmetry, each is drawn in one color on the right.

The coloring shown in Figure 2.4.3-A has three CW prototiles, $[F---f---]^2$, $[G---g----]^2$, and $[H---h----]^2$. Each of these has 2-fold rotational symmetry and appears in just one orientation.

2.5 The $(3^6; 3.3.6.6)$ -grid

Our curve-set of order 9 has maps

```
F |--> F++G++H++F++G---g++++f---F--f---F++G---g++++f---F
```

- f |--> f++++h---H---h++++g++++f---F---f
- G |--> G++H++F++G++H---h---H---h++++g---G
- g |--> g++++f---F---f++++h++++g---G---g
- H |--> H++F++G++H---h---H
- h |--> h++++g---G---g++++f++++h---H---h



Figure 2.4.3-B: Mutual self-similarity of the CW prototiles $[F---f---]^2$ (left, small copies blue), $[G---g---]^2$ (middle, small copies red), and $[H---h----]^2$ (right, small copies green).

The mutual self-similarity of the three tiles is shown in Figure 2.4.3-B. Observe how each tile appears in just one orientation. Borders are drawn between the curves, this helps to spot the borders between the small copies of the blue prototile on the left.

2.5 The (3⁶; 3.3.6.6)-grid

Here we use turns by 60 degrees $(2\pi/6)$.



Figure 2.5.0-A: Grid-graph of the $(3^6; 3.3.6.6)$ -grid (left) and an area drawing of the same (right).

The $(3^6; 3.3.6.6)$ -grid is shown in Figure 2.5.0-A. The grid has five classes of edges, denoted by the letters A ... E. The transitions are

```
A+C, B++D, C+A, D++E, E++B,
A-B, B-A, C--D, D--E, E--C,
DOE
```

Note that there are three transitions from D to E. The CCW prototiles are $[A+C+]^3$ and $[B++D++E++]^1$, and the CW prototiles are $[A-B-]^3$ and $[C--D--E--]^1$. We only use this minimal coloring of the grid.

2.5 The $(3^6; 3.3.6.6)$ -grid

A curve set of order 16

Our curve-set has order 16 and maps

```
A |--> A-B-A+C+A+C+A+C--D++E--C+A-B-A-B-A-B++D++E--C--D++E++B-A
```

- B |--> B-A-B++D--E++B++D--E++B++D--E--C+A-B
- C |--> C+A+C--D++E--C--D++E--C--D++E++B-A+C
- D |--> D++E--C--DOE++B++D--E++B-A-B-A-B++D
- E |--> E--C+A+C+A+C--D++E--C--D0E++B++D--E

It was designed so that the curves have mostly straight borders.

The mutual self-similarity of the five curves is shown in Figure 2.5.0-B. The shapes of the CCW prototiles are shown in Figure 2.5.0-C. The tiling of the plane by the five curves is shown in Figure 2.5.0-D.



Figure 2.5.0-B: Mutual self-similarity of the curves (left to right, top to bottom): A (small copies blue), B (small copies red), C (small copies green), D (small copies pale pink), and E (small copies olive).



Figure 2.5.0-C: Mutual self-similarity of the CCW prototiles $[A+C+]^3$ (left, small copies blue) and $[B++D++E++]^1$ (small copies red). Note that all but one of the prototiles $[B++D++E++]^1$ appear in (triangular) clusters of three. The two images colored by curves (right) may help to see the structure.



Figure 2.5.0-D: Tiling the plane with the curve-set, coloring as in grid.

A curve set of order 13 with just one non-constant symbol



Figure 2.5.0-E: First (left) and second iterate (middle) of the curve for the letter A (left), a similarly looking curve on the triangle grid (right).

The curve-set of order 13 has the following map for the letter $\mathtt{A},$ all other letters are constants.

It was created to look similar to the curve of order 13 named R13-15 from [2, Figure 3.1-A, p. 15] with map

F |--> F+F0F0F-F-F+F0F+F+F-F0F-F

For higher iterates the two curves are hard to tell apart as their limiting shapes are identical. The fourth iterate is shown in Figure 2.5.0-F. Iterates of the tile $[A+C+]^3$ are shown in Figure 2.5.0-G. The tile has 6-fold rotational symmetry in the limit. For higher iterates all but the blue and green regions (for the letter A) will vanish.

Indeed any curve on the minimal coloring of the triangle grid can be turned to a curve-set on the $(3^6; 3.3.6.6)$ -grid with all letters but A constant: In the map for the curve on the triangle grid, replace all + by p, all - by m, all 0 by n, all p by B++D++E--C, all m by B++D--E--C, all n by B++D0E--C, and all F by +A-, finally remove the leading + and the trailing -. This gives the map for A. In our example, the following command produces the map for A shown above (split into pieces for readability).



Figure 2.5.0-F: Self-similarity of the curve for the letter A.



Figure 2.5.0-G: Area drawings for the first (left) and second (right) iterates of the tile $[A+C+]^3$. Coloring by the letters and their orientation in the first iterate.

```
echo 'F+F0F0F-F-F+F0F+F+F-F0F-F' | \
  sed 's/+/p/g; s/-/m/g; s/0/n/g;' | \
  sed 's/p/B++D++E--C/g; s/m/B++D--E--C/g; s/n/B++D0E--C/g; s/F/+A-/g;' | \
  sed 's/^+//; s/-$//'
```

Note that dropping all but the A-edges in the grid (Figure 2.5.0-A on page 54) gives the triangle grid with minimal coloring.

2.6 The (3.4.4.6; 3.4.6.4)-grid

The (3.4.4.6; 3.4.6.4)-grid and its prototiles are shown in Figure 2.6.0-A (same as Figure 1.1.0-B on page 3). The prototiles are $[A++]^6$, $[B+++C++++D++++]^1$, $[E+++F+++]^2$, $[A--B--F--D---]^1$, and $[C--E--]^3$.

Turns are by 30 degrees $(2 \pi/12)$. The transitions are A++A, A---B, B++++C, B---F, C++++D, C--E, D++++B, D---A, E+++F, E--C, and F+++E, and F---D. As the pair of letters at start and end uniquely determine the turn, we can omit the turns in the maps to save space.



Figure 2.6.0-A: Grid-graph for the (3.4.4.6; 3.4.6.4)-grid (left) and its CCW (middle) and CW (right) prototiles.

A curve-set of order 7 with four constant letters

We give an example of a plane-filling curve-set where only two of the five prototiles are plane-filling. It has order 7 and just two non-constant maps:

- A |--> AAABFEFDABCECECDBFEFDAAA
- F |--> FDBFDBFECDBFECECECDBF

The maps for the letters $B,\,C,\,D,$ and E are constant. The motifs for A and F are shown in Figure 2.6.0-B.



Figure 2.6.0-B: Motifs for the curves \mathtt{A} (left) and \mathtt{F} (right).



Figure 2.6.0-C: First (left) and fourth (right) iterate of the prototile [A++]⁶.



Figure 2.6.0-D: First (left) and fifth (right) iterate of the prototile [E+++F+++]².



Figure 2.6.0-E: First (left) and fifth (right) iterate of the prototile $[A--B--F--D---]^1$.

Iterates of the prototiles $[A++]^6$, $[E+++F+++]^2$, and $[A---B---F--D---]^1$ are respectively shown in Figures 2.6.0-C, 2.6.0-D, and 2.6.0-E. The prototiles $[B++++C++++D++++]^1$ and $[C--E--]^3$ have all edges constant.



Figure 2.6.0-F: Mutual self-similarity of curves $\tt A$ (left) and $\tt F$ (right). Coloring by curves and orientation.

The mutual self-similarities of the curves A and F are shown in Figure 2.6.0-F. The

substitution matrix for this curve-set is

$$M = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 4 \\ 2 & 0 & 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 1 & 0 & 4 \\ 2 & 0 & 0 & 0 & 1 & 4 \\ 2 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

The curve for A (corresponding to the first column of the matrix) has dimension 2. The curve for F (last column) has dimension $2 \cdot \log_7 5 \approx 1.65417$, as the map for F contains five letters F (the lower right entry in the matrix) and no letters corresponding to curves with dimension 2. The constant letters B, C, D, and E do not contribute to the dimensions of A and F. Any map containing letters corresponding to a curve with dimension 2 gives a 2-dimensional curve. As the curve for A is the only 2-dimensional curve in this curve-set, the map for F cannot contain the letter A. It indeed does not, see the upper right entry of the matrix.

A curve-set of order 31



Figure 2.6.0-G: Top to bottom, left to right: motifs of A, B, C, D, E, and F. Start and end marked for each curve. Coloring by letters.

Our curve-set is of order 31 and has maps

- A |--> ABCDAABFEFDABCECECDBFEFDABCECDBFDBFDBCEFDBCECEFDABCDA
- B |--> BFEFDBFECECECDBFDAAAABCDAAB
- C |--> CEFDABCEFEFDAABCDAAABFEC
- D |--> DBFDAAAAABFDBCECECEFEFD
- E |--> ECDABFECDBFDBFEFDBCECECDABCDAABFDBFE
- F |--> FECECECDBFEFDABCDABFEF

The first iterates of the curves are shown in Figure 2.6.0-G.

A coloring by orientation of the *undirected* edges of the grid is shown in Figure 2.6.0-H. This coloring brings out the bands of squares in the grid. For the CW tiles in Figure 2.6.0-I this coloring is used.

The mutual self-similarities of the CCW tiles is shown in Figure 2.6.0-J. Line drawings of the motifs on the left, area drawings of the third iterates on the right.



Figure 2.6.0-H: Grid with edges colored by rotation modulo 180 degrees.



Figure 2.6.0-I: First iterates of the CW prototiles $[A---B---F--D---]^1$ (top left) and $[C--E--]^3$ (to right). Third iterates of the same at the bottom. Coloring by orientation modulo 180 degrees.



Figure 2.6.0-J: Mutual self-similarity of the CCW prototiles $[A++]^6$ (top, small copies blue), $[B+++C++++D++++]^1$ (middle, small copies red), and $[E+++F+++]^2$ (bottom, small copies green).

3 Curve-sets for grids with odd valencies



Figure 3.0.0-A: The tiling of the plane with hexagons. The vertex configuration is (6^3) , three edges are incident to each point.

So far we gave curve-sets for grids where every point has even valency (number of incident edges).

In the hexagon grid shown in Figure 3.0.0-A every point has valency 3. Every time a curve visits a point it needs to leave. That is, we seemingly have the problem that at a point of odd valency we cannot avoid creating a dead end.

The solution is quite simple: edge bifurcation. We declare some edges to be antiparallel pairs of edges. Once all points have even valency we can work (almost) as before. For our examples we always turn all edges into double edges.

Any such pair of *double edges* creates a *digon*, which has to be considered a prototile. *do* At both ends of a digon a turn by 180 degrees (U-turn) can appear, we reserve the *dig* symbol ! to specify U-turns in the Lindenmayer system. In our examples these digon tiles are always CW.

double edge digon

One can also create double edges where it is not strictly needed, we will do so for the square grid and the triangle grid.



Figure 3.0.0-B: First iterate of a curve on the triangle grid with double edges: line drawing (left) and area drawing (right). Areas are on the left of the edges.

In the area drawings the area assigned to an edge will be on its left, see Figure 3.0.0-B (compare to Figure 1.2.0-B on page 6).

3.1 The hexagon grid



Figure 3.1.0-A: The hexagon grid with double edges (left). In the area drawing (right) each edge corresponds to the triangular area left of it, coloring by orientation.

In the hexagon grid with double edges is shown in Figure 3.1.0-A. Each edge is responsible for an area to its left,

There is just one edge class; we use the letter A for it.

Here we use turns by 60 degrees $(2\pi/6)$.

The transitions are A+A, A-A, and A!A. There is one CCW prototile $[A+]^6$, a lattice tile, and one CW prototile $[A!]^2$, a digon tile.

A curve of order 25



Figure 3.1.0-B: Border of the CCW tile $[A+]^6$ (left) and tile filled with a curve of order 25 (right). First iterates, coloring by orientation of the curves.

Creating a curve is quite easy. Start with a (self-avoiding) curve with 2-fold symmetry. The symmetry is necessary to make the digon tile work.

Now we have a CCW tile $[A+]^6$ which is not yet filled, shown in Figure 3.1.0-B (left). The unfinished curve has the (incomplete) map

A |--> A+A-A+A+A+A+A-A-A-A-A+A-A

Filling the tile is possible in more than one way. The completion shown on the right corresponds to a curve with the map

Area drawings of the tiles are shown in Figure 3.1.0-C (CCW) and Figure 3.1.0-D (CW, digon).



Figure 3.1.0-C: Area drawings of the first (left) and third (right) iterate of the tile $[A+]^6$.



Figure 3.1.0-D: Area drawings of the first (left) and third (right) iterate of the digon tile $[A!]^2$.

A curve of order 109



Figure 3.1.0-E: First iterate of a curve of order 109. It starts at the bottom and ends at the top. Line drawing (left) and area drawing (right).

The motif of a curve of order 109 with map

is shown in Figure 3.1.0-E

The CCW tile is shown in Figures 3.1.0-F and 3.1.0-G, the CW (digon) tile in Figure 3.1.0-H.



Figure 3.1.0-F: First iterate of the CCW prototile $[A+]^6$. Line drawing (left) and area drawing (right). Coloring by orientation of the curves.



Figure 3.1.0-G: Second iterate of the CCW prototile $[A]^6$, decomposed into 109 smaller copies of its first iterate.



Figure 3.1.0-H: Second iterate of the digon tile $[A!]^2$ (left), decomposed into 109 smaller copies of itself (first iterate) appearing in three orientations (right).

A curve of order 19



Figure 3.1.0-I: First (left) and second (right) iterates of the curve. Starts and ends are marked. Coloring by orientation of the edges in the motif.

Figure 3.1.0-I shows the motif and second iterate of a curve of order 19 with map

The fourth iterate is shown in Figure 3.1.0-J. Now compare to Figure 2.4.1-M on page 50, the images look identical.

Indeed, take the map for the non-constant letter A for the curve of Figure 2.4.1-M,

Remove all letters B:

Normalize turns:



Figure 3.1.0-J: Fourth iterate, coloring by orientation of edges in the motive.

Finally replace --- by !, they are U-turns. This gives the map of our curve.

This is not a surprise. Reducing the edges B in the (3.4.6.4)-grid (the red ones in Figure 2.4.1-A on page 45) to zero gives the hexagon grid with double edges. The lozenge shaped areas in Figure 3.1.0-J are the digon tiles $[A!]^2$.

A coloring with three edge classes



Figure 3.1.0-K: A coloring of the hexagon grid with three colors. Line drawing (left) and area drawing (right).

We use the coloring with three colors (edge classes) shown in Figure 3.1.0-K. The CCW tile is $[A+B+C+]^2$, it is a lattice tile. The CW tiles are $[A!]^2$, $[B!]^2$, and $[C!]^2$.

The motifs for a curve-set of order 13 with maps

```
A |--> A+B+C+A+B!B+C+A-C!C+A-C+A+B-A
```

```
 B | --> B+C+A+B+C!C+A+B!B-A+B+C-B
```

```
C |--> C+A+B+C+A!A+B!B+C+A-C
```

```
is shown in Figure 3.1.0-L.
```

The mutual self-similarity of the CW tiles is shown in Figure 3.1.0-M.

3.2 The square grid



Figure 3.1.0-L: First iterates of the curves A (left), B (middle), and C (right). Each curve starts at the bottom and ends at the top right as indicted.



Figure 3.1.0-M: Mutual self-similarity of the CW tiles $[A!]^2$ (left, small copies blue), $[B!]^2$ (middle, small copies red), and $[C!]^2$ (right, small copies green).

3.2 The square grid



Figure 3.2.0-A: Grid-graph of the square grid with double edges, line drawing (left) and area drawing (right). Coloring by the orientations of the edges.

Though not needed, we can use double edges for the square grid, see Figure 3.2.0-A.

There is just one edge class.

Here we use turns by 90 degrees $(2\pi/4)$.

The transitions are A+A, AOA, A-A, and A!A. The CCW prototile is $[A+]^4$, it is a lattice tile. The CW (digon) tile is $[A!]^2$.

A curve of order 4

The only curve of order 4 is well-known, see [19, Section 4.5, pp.60-61] or [5, Item 5.3, Section A.5, p.243]. We start with an empty 2×2 square, see Figure 3.2.0-B (left).

3.2 The square grid



Figure 3.2.0-B: Border of the CCW tile $[A+]^4$ (left) and the tile filled with a curve of order 4 (middle). Second iterate of the tile (right).

There is just one way to fill the square (middle). The map of the resulting curve is

 $A \mid --> A+A!A+A$

The second iterate of this tile $[A+]^4$ is shown on the right. The limiting shape of the curve is a triangle.



Figure 3.2.0-C: The curve rendered to traverse the points of the (4.8.8)-grid.

The curve can be rendered so that it traverses the points of the (4.8.8)-grid as shown in Figure 3.2.0-C and [18, Figure 5, p. 428]. Here is a recipe for this rendering from our curve: in the word produced by the Lindenmayer system replace, in this order, all + with +F+, all A with +F-F-F+, finally all ! with --F--. For drawing the curve, use turns by 45 degrees $(2\pi/8)$.

A curve of order 5



Figure 3.2.0-D: Border of the CCW tile $[A+]^4$ (left) and the tile filled with a curve of order 5 (right).

A curve of order 5 with map

A |--> A+AOA!A+A

was given in [16, lower left in Figure 2.5, p. 16]. The first iterate of its CCW tile $[A+]^4$ is shown in Figure 3.2.0-D.

The sixth iterates of both tiles $[A+]^4$ and $[A!]^2$ are shown in Figure 3.2.0-E. Note that the shape of the tile $[A!]^2$ is that of the curve of order 5 with map



Figure 3.2.0-E: Sixth iterates of the tiles $[A+]^4$ (left) and $[A!]^2$ (right).



Figure 3.2.0-F: First (left) and sixth (right) iterate of a curve of order 5 on the square grid without double edges.

$F \mid --> F+F+F-F-F$

on the square grid without double edges, see Figure 3.2.0-F. This curve is the only one of order 5 on the square grid with minimal coloring.

A curve of order 17



Figure 3.2.0-G: Border of the CCW tile $[A+]^4$ (left), tile filled with an order 17 curve, line drawing (middle) and area drawing (right). First iterates.

Creating curves manually is as straightforward as for the hexagon grid, see Figure 3.2.0-G for a curve of order 17 with map

A |--> A+A!A+A+AOAOA!AOA+A+A+A!A+A!AOA+A

Fourth iterates of the tiles and the self-similarity of the tile $[A!]^2$ are shown in Figure 3.2.0-H. That the small copies on the right appear in two orientations should be no surprise, $[A!]^2$ lies along the edge pairs and those appear in horizontal and vertical orientation.



Figure 3.2.0-H: Fourth iterates of the tiles $[A+]^4$ (left) and $[A!]^2$ (middle). Self-similarity of the digon tile $[A!]^2$ (right), small copies appear in two orientations.

A curve of order 85



Figure 3.2.0-I: Motif of the curve. Line drawing (left) and area drawing (right). The curve starts at the upper left and ends at the lower right.

This curve of order 85 has the map

The motif is shown in Figure 3.2.0-I.

The self-similarity of the tile $[A+]^4$ is shown in Figure 3.2.0-J.


Figure 3.2.0-J: The second iterate of the CCW prototile $[A+]^4$ (left), decomposed into 85 small copies of its first iterate (right).

3.3 The triangle grid

The triangle grid with double edges is shown in Figure 3.3.0-A. There are now six orientations of the edges in the grid.

Turns are by 60 degrees $(2\pi/6)$ and all six transitions are possible, AOA, A+A, A++A, A!A, A-A, and A--A.

There is just one edge class. The CCW tile is $[A++]^3$, the CW tile is the digon $[A!]^2$.



Figure 3.3.0-A: Grid-graph of the triangle grid with double edges, line drawing (left) and area drawing with colors indicating orientation (right).

A curve of order 4

Our first example appears (in some disguise) in [16, Figure 2.9, p. 19], where it is attributed to [20]. The maps of the curve given are

```
A |--> AF+F+AF-F-F-AF-F+F+F-F+F+A
F |--> F
```



Figure 3.3.0-B: First iterate of a curve of order 4 (top left). First (bottom left) and sixth (middle) iterate of its CCW tile $[A++]^6$. Sixth iterate of the CW (digon) tile $[A!]^2$ (right). Coloring by orientation of the curves.

The letters A are removed before drawing. We remove all letters F and normalize the turns. to obtain the arguably most simple curve on this grid, see Figure 3.3.0-B.



Figure 3.3.0-C: The curve, rendered as suggested by Szilard and Quinton: self-similarity (left) and the CCW tile (right).

Renderings as suggested by Szilard and Quinton are shown in Figure 3.3.0-C. The curves cover all points on the hexagon grid. The axiom for the tile on the right is $[AF+F+]^3$.

A curve of order 3

This curve has the map

A |--> A++A--A

which is just that of the terdragon, whose map is usually given as

F |--> F+F-F

for turns by 120 degrees $(2\pi/3)$.

Here we took A+A-A which has 2-fold symmetry as the tile border and notice nothing can (or should) be filled into the interior of the first iterate of the tile $[A++]^6$, see Figure 3.3.0-D. Note how the area drawing on the right leaves half of the interior empty. This is because the curve is just half of the terdragon.

The CW (digon) tile covers the area of the terdragon, see the middle of Figure 3.3.0-E. The image on the right for the axiom A++A++A!A--A--A has the same shape as



Figure 3.3.0-D: First iterate of a curve of order 3 (top left) and its CCW tile $[A++]^3$ (bottom left). The third iterate of the tile drawn using lines (middle) and as area drawing (right). Coloring by orientation of the curves.



Figure 3.3.0-E: Third iterate of the CW tile $[A!]^2$ with U-turns marked by the two disks (left) and area drawing of the same (middle). The area drawing for the axiom A++A++A!A--A--A (right).

the tile for the terd ragon (with axiom $[\texttt{F+}]^3,$ a lattice tile), compare to the right of Figure 3.3.0-D.

Such a bisection of curves can be done for curves with 2-fold symmetry. Another example would be the (only) curve of order 4 with map

F |--> F+FOOF-F

A curve of order 61

Figure 3.3.0-F: Motif of a curve of order 61 (left), the curve starts at the lower left and ends at the upper right. Area drawing of the same (right).

The motif of the curve is shown in Figure 3.3.0-F, its CCW tile in Figure 3.3.0-G, and its CW (digon) tile in Figure 3.3.0-H.



Figure 3.3.0-G: The third iterate of the CCW prototile $[A++]^3$ (right), decomposed into 61 copies of its second iterate (left), appearing in 2 orientations.



Figure 3.3.0-H: The third iterate of the prototile $[A!]^2$ (left), decomposed into smaller copies of itself appearing in three orientations (right).

A coloring with two edge classes



Figure 3.3.0-I: Coloring the triangle grid with double edges using two colors. Line drawing (left) and area drawing (right).

The coloring (see Figure 3.3.0-I) is such that we have two CCW prototiles, $[A++]^3$ and $[B++]^3$. The color is switched over all edges, so the CW (digon) tile is $[A+B]^1$.

We need two CCW tiles that fit together. The borders of the tiles chosen are shown next to each other in Figure 3.3.0-J. Filling them gives the curves. The resulting curve-set has order 13 and maps

```
A |--> A++A!B++B!A++A++A!BOB+A
B |--> B++B!A++A!B++B!A+B++B++B!A+B!AOA+B
```



Figure 3.3.0-J: Borders of the CCW tiles $[A++]^3$ (blue) and $[B++]^3$ (red), drawn side by side (left). Filling the tiles gives the curves (right).

The self-similarity of the CW (digon) tile $[A!B]^1$ is shown in Figure 3.3.0-K. The mutual self-similarity of the CCW tiles in Figure 3.3.0-L.

In Figure 3.3.0-E on page 75 we have split the terdragon into halves in the minimal coloring. With the coloring here we can split any curve from the (minimally colored) triangle grid (with no double edges) into halves.

The digon tile [A!B!]¹ of such a curve of order 13 with maps

```
A |--> AOA++AOAOA--A++A--A-AOA++A++A--A
```

B |--> B++B--B--B0B++B++B--B++B0B0B--B0B

is shown in Figure 3.3.0-M. We have seen half of this tile (that is, curve A) in Figure 3.0.0-B on page 63. This is the splitting of the curve shown in Figure 1.2.0-A on page 5.

The same splitting technique works for the square grid with double edges and the coloring into two colors giving a chessboard pattern. We'll do the equivalent for the trihex grid next.



Figure 3.3.0-K: Area drawing of the first iterate of the CW (digon) tile $[A!B]^2$ (left). The third iterate (middle), decomposed into 13 copies of the second iterate, each colored by orientation (right).



Figure 3.3.0-L: Mutual self-similarity of the CCW tiles: $[A++]^3$ (left, small copies in blue) and $[B++]^3$ (right, small copies in red).



Figure 3.3.0-M: First (left) and second (middle) iterate of a digon tile, colored by letters. Second iterate decomposed into first iterates, colored by orientations (right).

3.4 The trihexagonal grid



Figure 3.4.0-A: The trihex grid with double edges, line drawing (left) and area drawing (right).

Here we use turns by 60 degrees $(2\pi/6)$.

The minimal coloring of the grid has two edge classes, see Figure 3.4.0-A.

The CCW prototiles are $[F+]^6$ (blue hexagons) and $[G++]^3$ (red triangles), The CW (digon) prototile is $[F!G]^1$. The transitions are F+F, FOG, F--F, F!G, G++G, GOF, G-G, and G!F.

A curve-set of order 13



Figure 3.4.0-B: From left to right: curve of order 13 on the trihex grid, its area drawing, the digon tile $[F!G]^1$ on the grid with double edges, and its area drawing.

We start with the curve of order 13 on the trihex grid without double edges with map

 $F \quad | --> \quad F+F+F+F+F--F+F+F--F-F+F+F--F$

It is shown on the left in Figure 3.4.0-B.

For the grid with double edges we need a curve G that is the reversal of the curve $F\colon$ its map is obtained by reversing the map for F and changing all + into - and all - into +:

 $\mathsf{G} \hspace{0.1in} | \hspace{0.1in} - \hspace{0.1in} > \hspace{0.1in} \mathsf{G} + \hspace{0.1in} \mathsf{G} - \hspace{0.1in} \mathsf{G} - \hspace{0.1in} \mathsf{G} + \hspace{0.1in} \mathsf{G} - \hspace{0.1$

Now $[F!G]^1$ is already the CW (digon) tile, nothing needs to be filled in, see the two images on the right in the figure.

Figure 3.4.0-C shows the tiling of the plane by (second iterates) of the digon tiles. Alternatively, it shows the tiling of the plane by the curve F we started with, on the grid without double edges.

3.4 The trihexagonal grid



Figure 3.4.0-C: Tiling the plane by digon tiles $[F!G]^1$. Coloring by orientation of tiles, not distinguishing the two curves in any tile.

A curve-set of order 9 with the Koch snowflake as CCW tiles



Figure 3.4.0-D: First (left) and fourth (right) iterate of the CCW tile $[F+]^6$ and $[G++]^3$ drawn next to each other.

We can create a curve-set such that the CCW tile $[G^{++}]^3$ has 6-fold symmetry. The most simple example is the curve-set of order 9 with maps

```
F |--> F+F+F!G++G+FG!F--F+F+F
G |--> G++G!F+F!G++G+G-G
```

whose CCW tiles are shown in Figure 3.4.0-D. The shape of both tiles is that of the Koch snowflake, even having reflection symmetry.

Another curve-set with two 6-fold symmetric tiles

The borders of the CCW tiles of a curve-set are shown in Figure 3.4.0-E. The maps for the borders are

F |--> F+F+F+F--F--F+F+F+F--F

 $\mathsf{G} \quad | \text{-->} \quad \mathsf{G}\text{++}\mathsf{G}\text{-}\mathsf{G}\text{-}\mathsf{G}\text{-}\mathsf{G}\text{+}\mathsf{+}\mathsf{G}\text{+}\mathsf{+}\mathsf{G}\text{-}\mathsf{G}\text{-}\mathsf{G}\text{-}\mathsf{G}$

Again both tiles have 6-fold symmetry. They do not have the reflection symmetry like the Koch snowflake, though. This can be seen in Figure 3.4.0-F.



Figure 3.4.0-E: Borders of the CCW tiles, drawn next to each other. Tile $[F+]^6$ is the bigger one, tile $[G++]^3$ the smaller one. Line drawing (left) and area drawing (right).



Figure 3.4.0-F: Third iterates of the CCW prototiles: tile $[\tt F+]^6$ colored blue, tile $[\tt G++]^3$ colored red.

3.5 The (4.8.8)-grid

Here we use turns by 45 degrees $(2\pi/8)$.



Figure 3.5.0-A: The (4.8.8)-grid with double edges, line drawing (left, blue for A, otherwise red) and area drawing (right, blue for A, red for B, green for b).

We turn all edges of the (4.8.8)-grid into double edges as shown in the left in Figure 3.5.0-A. There are three edge classes. Letters A lie between octagons, letters b and B lie between a square and an octagon.

All edges are assigned an area left of them. The areas for b fill the squares, the areas

3.5 The (4.8.8)-grid

for A and B fill the octagons, see the right of Figure 3.5.0-A. The CCW prototiles are $[A+B+]^4$ and $[b++]^4$, CW prototiles are $[A!]^2$ and $[b!B!]^1$. The transitions are A+B, A!A, A-b, b++b, b-A, b!B, B+A, B--B, and B!b.



Figure 3.5.0-B: Area drawing of the grid colored by orientation of edges.

Figure 3.5.0-B shows an area drawing of the grid where the colors correspond to the orientation of the edges.

A curve of order 5 with two constant letters



Figure 3.5.0-C: From left to right: motif as line and as area drawing, second and third iterates as area drawing.

The curve has order 5 and maps

```
A |--> A-b++b++b+B+A+B--B+A!A+B+A
b |--> b
B |--> B
```

Its first, second, and third iterates are shown in Figure 3.5.0-C. The coloring used in the area drawing is by orientation of the edges, as in Figure 3.5.0-B. Note how all but the green and blue areas (corresponding to the letter A) vanish with higher iterates.

Iterates of the CW (digon) tile are shown in Figure 3.5.0-D. It looks suspiciously like what is shown in Figure 3.2.0-F on page 71. Indeed, by dropping letter b and B and normalizing turns one obtains the map

A |--> A++AOA!A++A

This is the map for the curve of order 5 on the double-edged square grid from Section 3.2 on page 69. Not a surprise, dropping edges b and B in our grid gives the square grid with double edges.



Figure 3.5.0-D: The CW tile $[A!]^2$, first (left), second (middle), and fourth (right) iterate. Coloring by the orientations of the edges in the first iterate.





Figure 3.5.0-E: Motifs for curves A (left), b (middle), and B (right).

The first iterates for a curve-set of order 13 with maps

A |--> A+B!b++b++b-A+B+A!A+B+A+B!b-A+B+A

```
b |--> b++b++b!B+A-b
```

B |--> B+A+B+A+B!b++b++b-A+B!b-A+B--B+A+B

are shown in Figure 3.5.0-E. Start and end points are marked. Figure 3.5.0-F shows how the curves tile the plane.



Figure 3.5.0-F: Tiling the plane by the curves, coloring by orientation.

3.6 The (3.12.12)-grid



Figure 3.6.0-A: The (3.12.12) grid with double edges. Line drawing with letters A blue, b and B red (left). Area drawing with letters A blue, B red, and b green (right).

The grid is shown in Figure 3.6.0-A. There are three edge classes.

Here we use turns by 30 degrees $(2\pi/12)$. The prototiles are $[A+B+]^6$ and $[b++++]^3$ (CCW), $[A!]^2$ and $[b!B!]^1$ (CW). The transitions are A+B, A-b, A!A, B+A, B----B, B!b, b++++b, b-A, and b!B. The letters at start and end of the transitions determine the turn uniquely, so we can omit them in the maps.

Our curve-set is of order 27 and has maps

- A |--> AB!bbbAB!bbbA!ABA!ABABABA!AB!bbbABABA
- B |--> BABABB!bbbABABB!bbbABB!bbbA!ABABABB!bbbABABAB
- b |--> bb!BAbbb!BAb

We left out the turns except for the U-turns A!A, B!b, and b!B.



Figure 3.6.0-B: First iterates of curve A (left), B (middle), and b (right). Coloring by axiom. Start and end of curves marked.



Figure 3.6.0-C: Area drawing of the third iterates of A (left), B (middle), and b (right).

The Motifs and third iterates are respectively shown in Figures 3.6.0-B and 3.6.0-C.



Figure 3.6.0-D: The CW (digon) tiles $[A!]^2$ (left) and $[B!b!]^1$ (right).

The CW (digon) tiles are shown in Figure 3.6.0-D.



Figure 3.6.0-E: The CCW tiles $[b++++]^3$ (left) and $[A+B+]^6$ (right). Coloring by letters in the axioms.

The CCW tiles, decomposed into curves, are shown in Figure 3.6.0-E. Note that the CCW tile $[b++++]^3$ has 6-fold symmetry.



Figure 3.6.0-F: The CCW tile $[b++++]^3$, decomposed into one copy of the CCW tile $[A+B+]^6$ (blue and red) and six copies of itself (green).

Figures 3.6.0-F and 3.6.0-G respectively show the tiles $[b++++]^3$ and $[A+B+]^6$, decomposed into smaller copies of each other. Coloring of the small copies is just as in Figure 3.6.0-E. The tiling of the plane by either the CCW tiles or the CW (digon) tiles (as in Figure 3.6.0-D) can also be gleaned from Figure 3.6.0-G.

3.6 The (3.12.12)-grid



Figure 3.6.0-G: The CCW tile $[A+B+]^6$, decomposed into copies if itself (blue and red) and copies of $[b++++]^3$ (green).

4 Curve-sets not fitting into our framework

Curve-sets as we defined them include all edge-covering families of plane-filling curves on grids we have seen before. Still, there is more. We present examples of edge-covering curves on grids that do not fit into our framework.

In the following we refer to the objects presented as curve-sets even though they are not curve-sets in the sense used so far.

4.1 First example



Figure 4.1.0-A: Motif for F (top left) and H (bottom left). Second iterate for F (middle) and H (right). Edges F are blue and edges H are red.

On the triangle grid consider the curve-set given by the maps

F |--> F+F-F-F0F+F+F-F0F

H |--> H+F-F-F+HOH

Turns are by 120 degrees $(2\pi/3)$, these curves live on the triangle grid. Motif and second iterates are shown in Figure 4.1.0-A. Note that there are 12 letters F and 3 letters H in the productions, so our concept of an order of a curve-set does not apply here. Edges H appear only on a line.



Figure 4.1.0-B: Fourth iterates of the tiles $[F+]^3$ (left) and $[H+]^3$ (right).

Note that the transitions are not as we required them for our curve-sets: in the second iterate for H most of the transitions are F+F and F-F (from the map for F) but some

4.2 Second example

are F-H (from the map for H). Therefore curve H does not live on a grid coloring as the one we used.

The tile $[F+]^3$ is shown on the left in Figure 4.1.0-B (the tile $[F-]^3$ is identical, modulo a reflection). Curve F occupies *half* of the plane, its tile $[H+]^3$ is shown on the right. The tile $[H-]^3$ has an empty triangular interior, it is obtained by reflecting the three curves in the tile $[H+]^3$ over their straight borders.

4.2 Second example

This is actually an infinite family of examples.



Figure 4.2.0-A: Motif (left) and self-similarity (right) of a curve of order 7.

We start with a curve of order 7 on the triangle grid given by the map

F |--> F+F-F-F+F+F-F

Its motif and decomposition into 7 smaller copies of itself are shown in Figure 4.2.0-A. Now duplicate the map for F into one for the letter G. Then, in the production, replace the letter F by G at will. We use the maps

```
F |--> F+F-F-G+F+F-F
G |--> G+G-G-G+G+G-G
```

We have 6 letters F and 8 letters G in the production. So this curve-set does not fit into our framework.



Figure 4.2.0-B: First (top left), second (bottom left), and fifth (right) iterate of curve F. Coloring by individual edges, blue for F and red for G.

These two curves "work" because they are of the same shape. The production of G does not contain any letter F so the curve is exactly the thing we started with. Curve

4.2 Second example

F looks more interesting, see Figure 4.2.0-B: the grid coloring induced by this curve has definitely no translation symmetry.



Figure 4.2.0-C: First (top left), second (bottom left, rounded turns), and fourth (right) iterate of curve F. There is always a nonzero turn between successive edges G (red) but not always for F (blue).

We can do more by using for F a curve of the same shape as G (but not the identical sequence of turns):

F |--> FOF+F+G-F-FOF (changed) G |--> G+G-G-G+G+G-G

Compare the motifs (top left) of Figure 4.2.0-C with that of Figure 4.2.0-B! In our illustration we use rounded corners to make regions with many turns (red) visually distinguishable from regions with straight lines.

Acknowledgments

It is our pleasure to thank the following people for their support, criticism, corrections, and improvements. Michel Dekking, Marvin Dittrich, Daniel Fischer, Christoph Haenel, Edith Parzefall, Neil Sloane, and Jeffrey Ventrella.

Index of terms

 $\mathrm{map},\,4$ k-fold symmetry, 10 area drawing, 5 axiom, 4 CCW sense of rotation, 4 constant, 4curve-set, 6CW sense of rotation, 4 digon, 63double edge, 63edge class, 3edge-covering, 5folding curves, 11 fundamental region, 20 tile, 4grid, 3iterate, 4lattice tile, 4 Lindenmayer system, 4

minimal coloring, 10 motif, 5 mutual self-similarity, 6 order, 7 patch, 8 plane-filling, 5 production, 4 prototile, 4 refined coloring, 10 self-avoiding, 5 self-similarity, 5 substitution matrix, 7 tile, 4 transitions, 3 unique transition property, 3 valency, 5

REFERENCES

References

- [1] Mark Anthony Armstrong: Groups and Symmetry, Springer-Verlag, (1988). 8
- [2] Jörg Arndt: Plane-filling curves on all uniform grids, arXiv:1607.02433
 [math.CO], (8-July-2016), URL: http://arxiv.org/abs/1607.02433. 4, 6, 7, 17, 20, 25, 37, 45, 56
- [3] Jörg Arndt, Julia Handl: Plane-filling Folding Curves on the Square Grid, Proceedings of Bridges 2018: Mathematics, Art, Music, Architecture, Education, Culture, Stockholm, (2018). URL: http://archive.bridgesmathart.org/ 2018/bridges2018-179.html. 20
- Jörg Arndt, Julia Handl: Puzzling plane-filling curves, Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture, Linz, (2019). URL: http://archive.bridgesmathart.org/2019/bridges2019-131. html.10, 16
- [5] Michael Bader: Space-Filling Curves, Springer-Verlag, (2013). 69
- [6] Aidan Burns: Fractal Tilings, The Mathematical Gazette, vol. 78, no. 482, pp. 193–196, (July-1994). 45
- [7] Chandler Davis, Donald E. Knuth: Number representations and dragon curves, in: Selected Papers on Fun and Games, CSLI Publications, pp. 571–614, (2011).
 5
- [8] F. M. Dekking: Iterated paperfolding and planefilling curves, Report 8126 of the Mathematical Institute, Katholieke Universiteit, Nijmegen, The Netherlands, pp. 1–16, (August-1981). 14
- [9] Michel Dekking: Paperfolding morphisms, planefilling curves, and fractal tiles, Theoretical Computer Science, vol. 414, no. 1, pp. 20-37, (January-2012). URL: http://dx.doi.org/10.1016/j.tcs.2011.09.025. 11, 12
- [10] Daniel Fischer: Entwicklung eines Algorithmus zur Suche von flächenfüllenden Kurven auf uniformen zweidimensionalen Gittern, master's thesis, Technische Hochschule Nürnberg, (2020). 14, 23, 35
- [11] Branko Grünbaum, Geoffrey C. Shephard: Tilings and Patterns, Freemann, NY, (1986). 3
- [12] Roger A. Horn, Charles R. Johnson: Matrix Analysis, second edition, Cambridge University Press, (2013). 15
- [13] Rudolf Lidl, Günter Pilz: Applied Abstract Algebra, second edition, Springer-Verlag, (1997). 8
- [14] Benoit B. Mandelbrot: The Fractal Geometry of Nature, W. H. Freeman, (1982). 5
- [15] Heinz-Otto Peitgen, Hartmut Jürgens, Dietmar Saupe: Chaos and Fractals, Springer-Verlag, (2004). 5
- [16] Przemysław Prusinkiewicz, James Hanan: Lindenmayer Systems, Fractals, and Plants, Lecture Notes in Biomathematics, vol. 79, Springer-Verlag, (1989). 70, 73
- [17] P. Prusinkiewicz, A. Lindenmayer: The Algorithmic Beauty of Plants, Springer-Verlag, (1990). URL: http://algorithmicbotany.org/papers/#abop. 4
- [18] Hans Sagan: Some reflections on the emergence of space-filling curves: the way it could have happened and should have happened, but did not happen, Journal of the Franklin Institute, vol. 328, no. 4, pp. 419–430, (1991). 70
- [19] Hans Sagan: Space-Filling Curves, Springer-Verlag, (1994). 69
- [20] A. Szilard, R. Quinton: An Interpretation for D0L-systems by computer graphics, The Science Terrapin, vol. 4, pp. 8–13, (1979). 73