Edge-covering plane-filling curves on grid colorings: a pedestrian approach

Jörg Arndt, Julia Handl

Computers in Scientific Discovery 10
Kortrijk, Belgium, May 2024

Abstract

We describe families of plane-filling curves on any edge-to-edge tiling of the plane with regular polygons and finitely many classes of edges. We indicate how to partition the minimal number of edge classes from the group G of symmetries of the tiling into refined colorings of the tiling, corresponding to finite subgroups of G. Each grid coloring corresponds to a family of plane-filling curves which we call curve-sets. Our exposition is driven by illustrated examples.

Contents

1 Recap: edge-covering curves

1.1 A motif with 13 eges

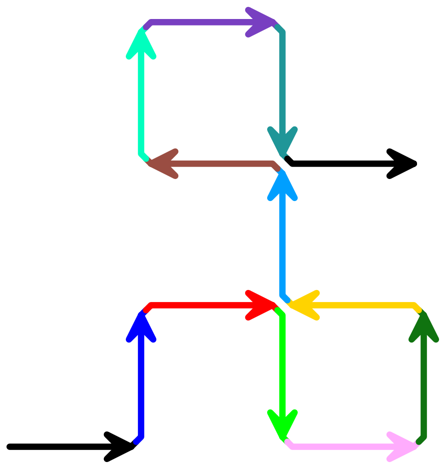

Figure 1.1-A: Motif of a curve with 13 edges.

This curve can be described by the word `F+F-F-F+F+F+F-F+F-F-F-F+F`: interpret `F` as "draw a unit edge", + as "turn left by 90 degrees", and - as "turn right by 90 degrees". The turns are drawn slightly rounded.

We now replace each edge of this *motif* by the motif, keeping the (randomly assigned) colors.

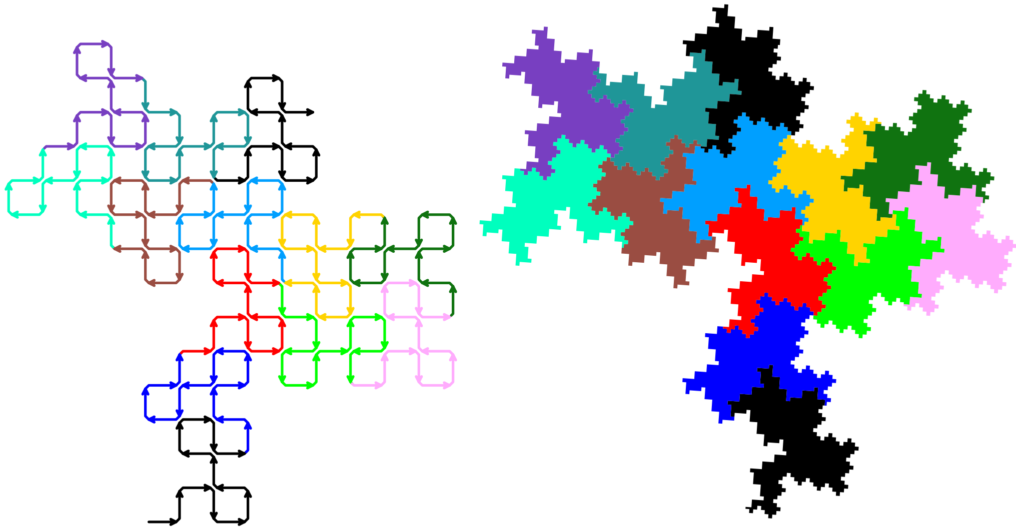

Figure 1.1-B: Second (left) and fourth (right) iterate of the curve.

In the second iterate (left) the curve does neither cross itself nor has double edges, it is *self-avoiding*. Replacing the edges in the second iterate with the motif gives the third. Doing that again gives the fourth, shown on the right.

The kth *iterate* is obtained by applying the substitution $\texttt{F} \mapsto \texttt{F+F-F-F+F+F+F-F+F-F-F-F+F}$ k times to the letter $\texttt{F}$.

The *self-similarity* is apparent in the fourth iterate: the curve can be decomposed into 13 smaller copies of itself.

1.2 The tile conditions

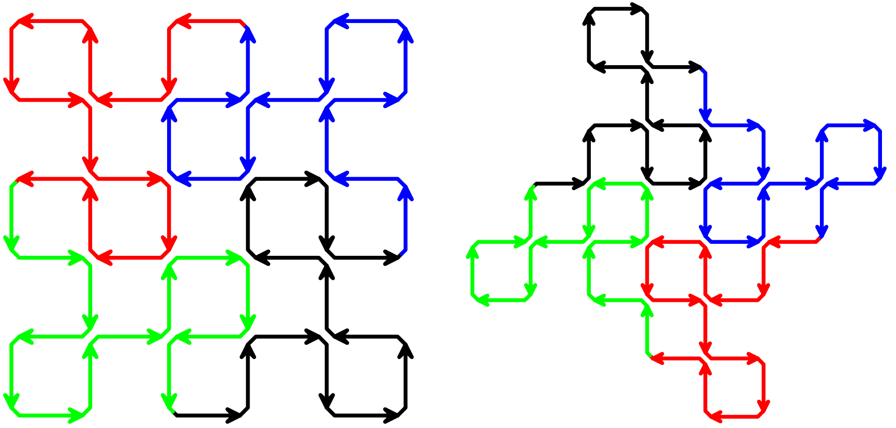

Figure 1.2-A: First iterates of the tiles Θ_{+1} (left) and Θ_{-1} (left).

The counterclockwise tile Θ_{+1} is obtained by replacing the edges of the CCW rectangle `F+F+F+F+` by the motif. For the clockwise tile Θ_{-1} start with the CW rectangle `F-F-F-F-`.

Theorems (Dekking): (1) If both tiles are self-avoiding then the curve is self-avoiding. (2) If the edges in the interior of both tiles are drawn then the curve is *plane-filling*.

Plane-filling: one can cover an arbitrarily large disk with (a high enough iterate) of the curve.

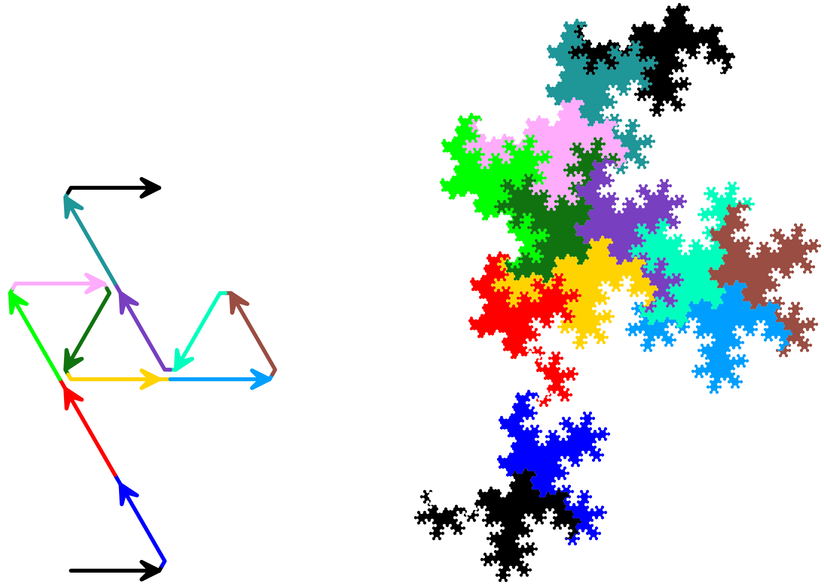

Figure 1.2-B: First (left, motif) and fourth (right) iterate of a curve described by the map `F` $\mapsto$ `F+F0F0F-F-F+F0F+F+F-F0F-F`. Turns are by 120 degrees.

The motif shown on the left uses turns by ± 120 degrees. The self-similarity is apparent on the right.

The *order* of a curve is the number of letters `F` in the map (of the Lindenmayer system). Both curves shown so far have order 13.

These curves are *edge-covering*: all edges in their interior are traversed once.

1.3 Self-similarity of the tiles

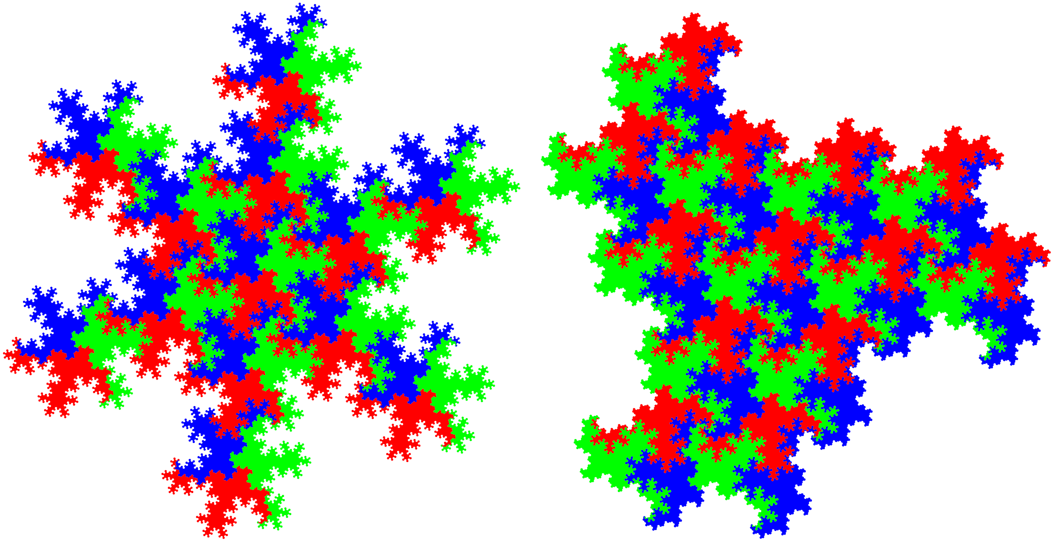

Figure 1.3-A: Fourth iterates of the tiles Θ_{+4} (left) and Θ_{-4} (right).

The fourth iterate of the tile Θ_{+4} (left), colored by the orientations of the curves contained. The tile can be decomposed into 13 smaller copies of itself (each containing one blue, one red, and one green curve).

The same is true for the Θ_{-4} (right).

That is, the tiles are self-similar as well!

1.4 Our curves live on (regular, directed) grids

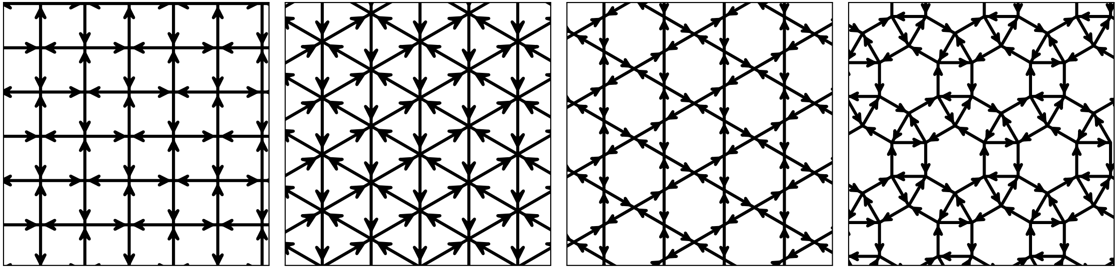

Figure 1.4-A: Directed grids, from left to right: square grid (4^4), triangle grid (3^6), trihexagonal grid $(3.6.3.6)$, and the $(3.4.6.4)$-grid.

The curve of our first example traverses the square grid, the curve of the second example traverses the triangle grid.

Both use just one letter (`F`) for drawing edges. The trihexagonal grid is the only remaining grid allowing for such curves.

This is because those three grids have just one *edge class*: two edges are in the same class if one can be moved to the other by the symmetries of the grid (translations and rotations). We disallow reflections.

Up to here things have been treated in
Jörg Arndt: Plane-filling curves on all uniform grids, arXiv:1607.02433 [math.CO], (8-July-2016), URL: http://arxiv.org/abs/1607.02433.

2 Grids with more than one edge class

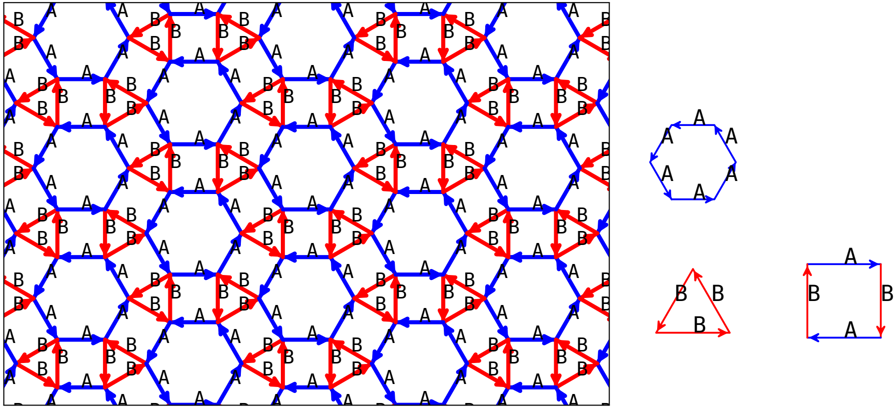

Figure 2.0-A: The minimal coloring of the $(3.4.6.4)$-grid (left). Its CCW prototiles [A++]^6 and [B++++]^3 (middle), and the CW prototile [A---B---]^2 (right).

The $(3.4.6.4)$-grid has two edge classes (left): The edges around the hexagons (letter A) are drawn blue and the edges around the triangles (letter B) are drawn red.

The letters will be used in the Lindenmayer systems.

The polygons of the grid give the *prototiles*. The sense of rotation does matter: the two prototiles for the square grid are [F+]^4 (CCW) and [F-]^4 (CW).

2.1 Curve-sets

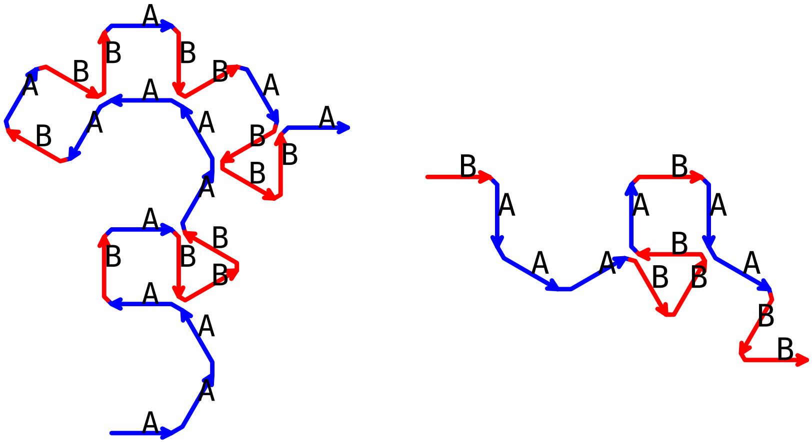

Figure 2.1-A: First iterates (motifs) of a curve-set on the $(3.4.6.4)$-grid. Motif for `A` (left) and for `B` (right).

With k classes of edges we get k curves working together. We call this a *curve-set*.

The two curves of the curve-set shown have the maps

```
A |--> A++A++A++A---B---A---B++++B++++B---A++A++A++A--- \
         B---A---B++++B---A---B++++B---A---B++++B++++B---A
B |--> B---A++A++A---B++++B++++B---A---B---A++A---B++++B
```

Here the turns (letters + and −) are by ± 30 degrees $(2\,\pi/12)$.

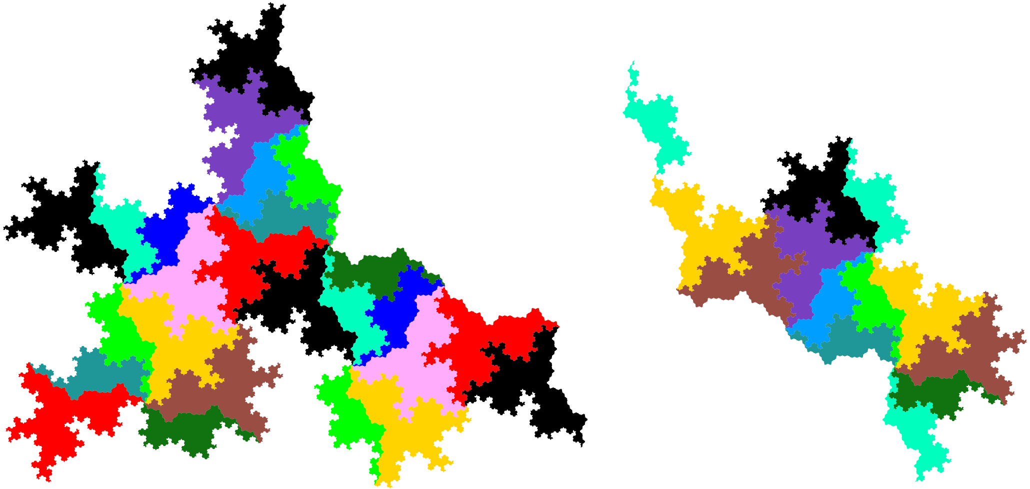

Figure 2.1-B: Fourth iterates of the curve-set, for `A` (left) and `B` (right). Coloring by orientation of the curves.

The *mutual self-similarity* of the two curves: both curves can be decomposed into smaller copies of each of them.

2.2 Prototiles and their mutual self-similarity

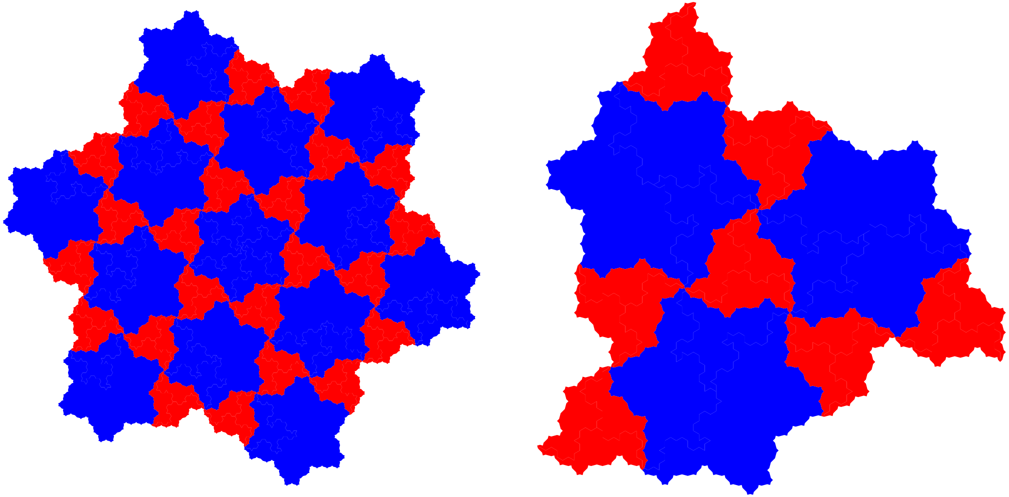

Figure 2.2-A: Shapes of the CCW prototiles $[\texttt{A++}]^6$ (left) and $[\texttt{B++++}]^3$ (right).

The two CCW prototiles (hexagon and triangle) are mutually self-similar. Together they tile the plane.

In general, all CCW prototiles of a curve-set are mutually self-similar and tile the plane. The same is true for the CW prototiles.

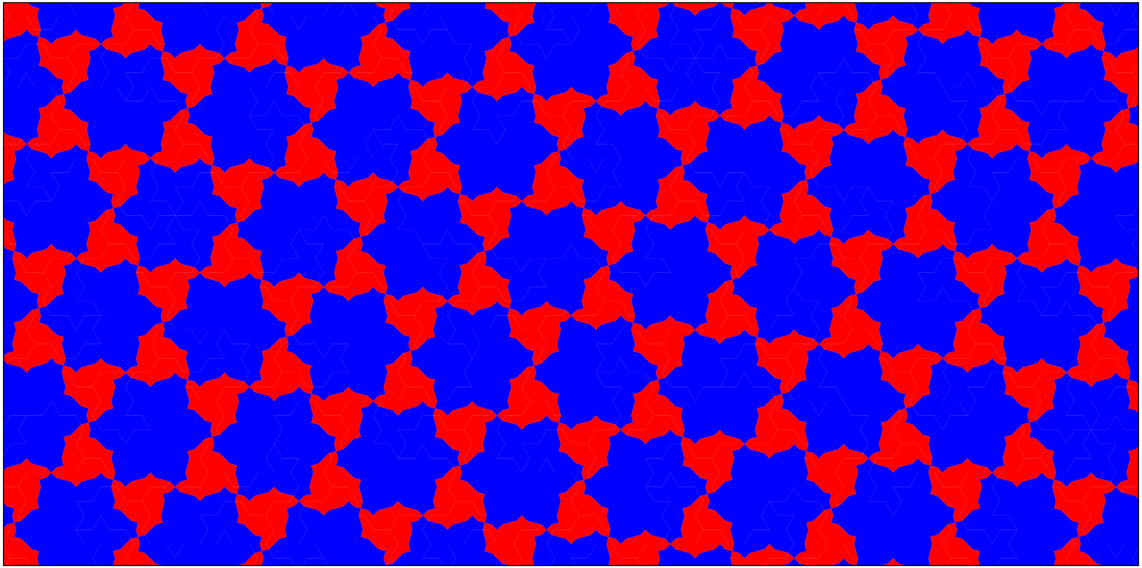

Figure 2.2-B: Together, the two CCW prototiles tile the plane.

As in the grid, the prototiles [B++++]^3 (red) appear in two orientations and the prototiles [A++]^6 (blue) in just one.

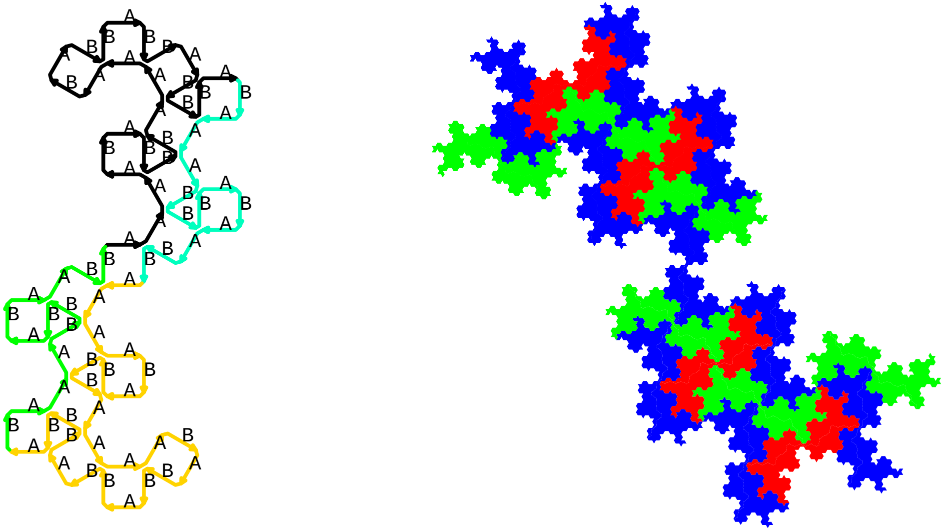

Figure 2.2-C: First (left) and third (right) iterate of the CW prototile [A---B---]^2.

There is just one CW prototile (the square), its is self-similar.

This prototile tiles the plane, appearing in three orientations.

2.3 Minimal grid colorings

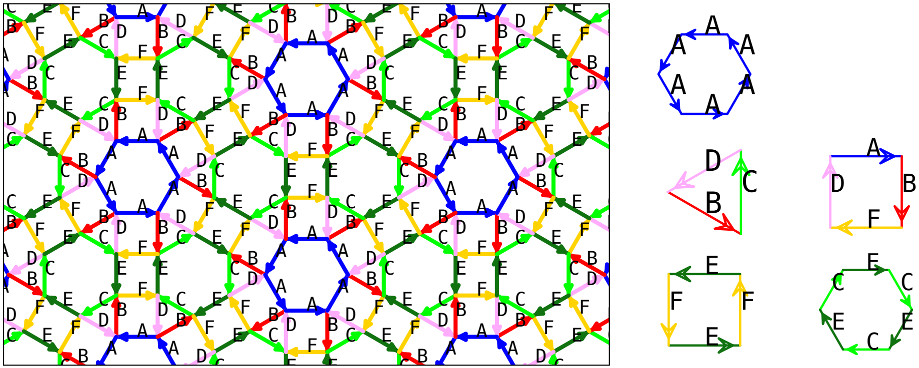

Figure 2.3-A: The $(3.4.4.6; 3.4.6.4)$-grid (left) and its CCW (middle) and CW (right) prototiles.

The six edge classes of a more complicated grid (left) and the prototiles (middle and right).

The drawing shows the *minimal coloring* of the grid; "minimal" as in the least amount of edge classes (equivalently, colors) is used.

We know how to create curve-sets on every grid: Dekking's tile conditions have to hold for all prototiles, that's it!

However, there is one more condition. The number of edges adjacent to each point must be even: for each edge leading to the point there must be another one leading away from it. We'll get rid of this apparent problem later.

3 Refined grid colorings

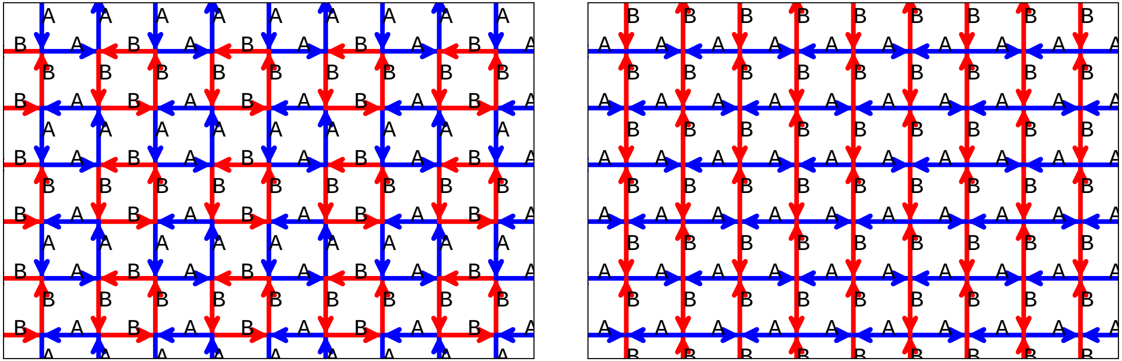

Figure 3.0-A: The two colorings of the square grid using two colors.

Refined grid colorings are colorings with more than the minimal number of edge classes. There are two such colorings of the square grid with two colors.

The conditions for such colorings are quite natural, see our paper.

The splendid news is that Dekking's tile conditions still work!

There is a little caveat, however: some curve-sets are not plane-filling anymore. Such "ugly" curve-sets can be discarded easily, so this isn't much of a problem.

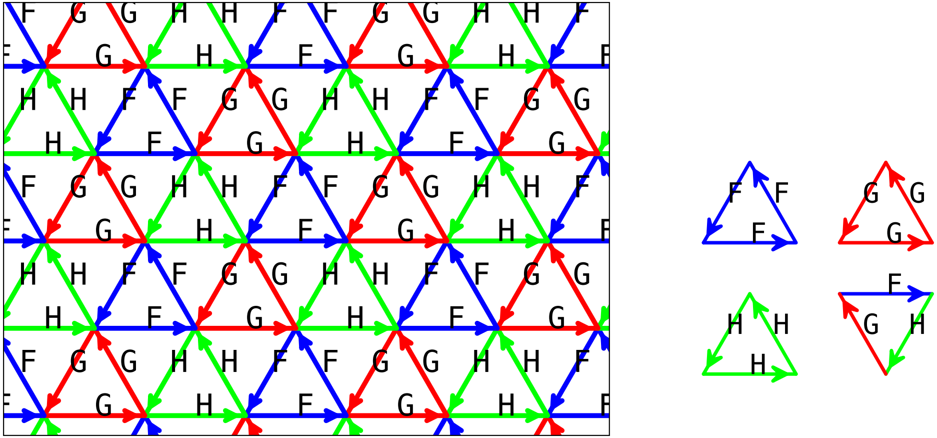

Figure 3.0-B: A coloring of the triangle grid with three colors (left) and its four prototiles $[\texttt{F+}]^3$, $[\texttt{G+}]^3$, $[\texttt{H+}]^3$, and $[\texttt{F-H-G-}]^1$ (right).

The coloring shown has the neat property that the three CCW prototiles ($[\texttt{F+}]^3$, $[\texttt{G+}]^3$, and $[\texttt{H+}]^3$) each have threefold rotational symmetry.

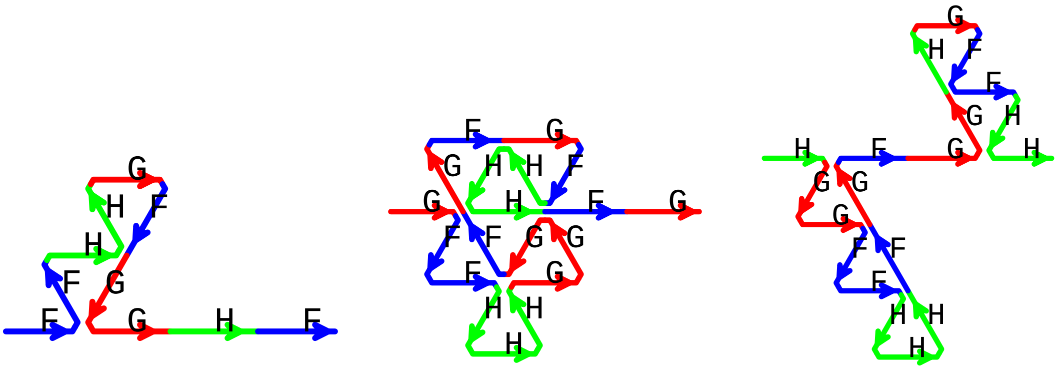

Figure 3.0-C: Motifs for the curves of a curve-set of order 16. From left to right, motifs for curve `F`, `G`, and `H`.

Here is a curve set on this grid coloring. Its maps are

```
F |--> F+F-H+H-G-F0G+G0H0F
G |--> G-F+F-H+H+H-G+G+G-F0G-F0G-F-H+H+H0F0G
H |--> H-G+G-F+F-H+H+H0F0G-F0G+G0H-G-F+F-H+H
```

The number of occurrences of the letter `F` in all three maps is 16. Same for the letters `G` and `H`. The is not a coincidence: the number of any letter in all maps has to be identical (Orbit-Stabilizer Theorem for group actions).

This observation helped to obtain the notion of an *order of a curve-set*. This curve-set has order 16.

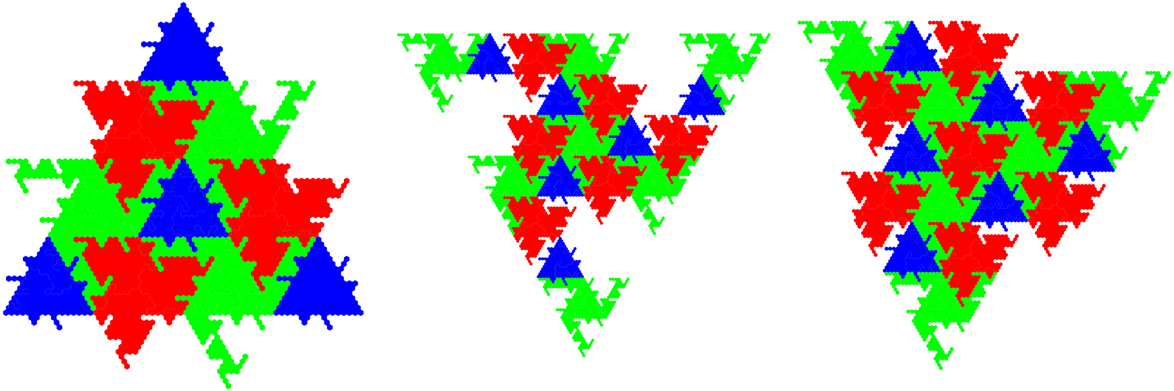

Figure 3.0-D: Mutual self-similarity of the CCW prototiles [`F+`]^3 (left), [`G+`]^3 (middle), and [`H+`]^3 (right).

Each of the three CCW prototiles can be decomposed into smaller copies of all three. The small copies of [`F+`]^3 (left), [`G+`]^3 (middle), and [`H+`]^3 (right) are respectively drawn in blue, green, and red.

Many (seriously: *many!*) more examples of curve-sets are shown in our paper Jörg Arndt, Julia Handl: Edge-covering plane-filling curves on grid colorings: a pedestrian approach, arXiv:2312.00654 [math.CO], (1-December-2023), URL: https://arxiv.org/abs/2312.00654.

4 Curve-sets for grids with odd valencies

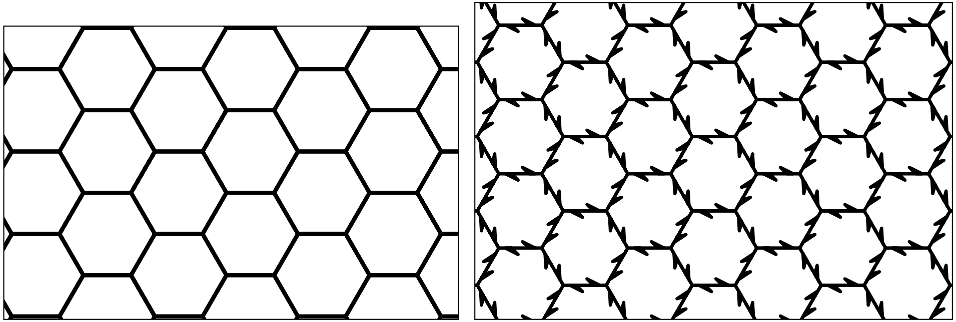

Figure 4.0-A: The tiling of the plane with hexagons. The vertex configuration is (6^3), three edges are incident to each point (left) and the same grid with double edges (right).

Each point of the honeycomb grid has three adjacent edges (left), seemingly an obstruction for edge-covering curves. There is a simple way out: consider each edge as a pair of anti-parallel edges (right).

Dekking's tile conditions *still* work!

There is just one edge class; we use the letter `A` for it. Here we use turns by 60 degrees $(2\pi/6)$.

There is one CCW prototile $[\texttt{A+}]^6$ (hexagon), and one CW prototile $[\texttt{A!}]^2$ (digon). The letter `!` denotes a U-turn.

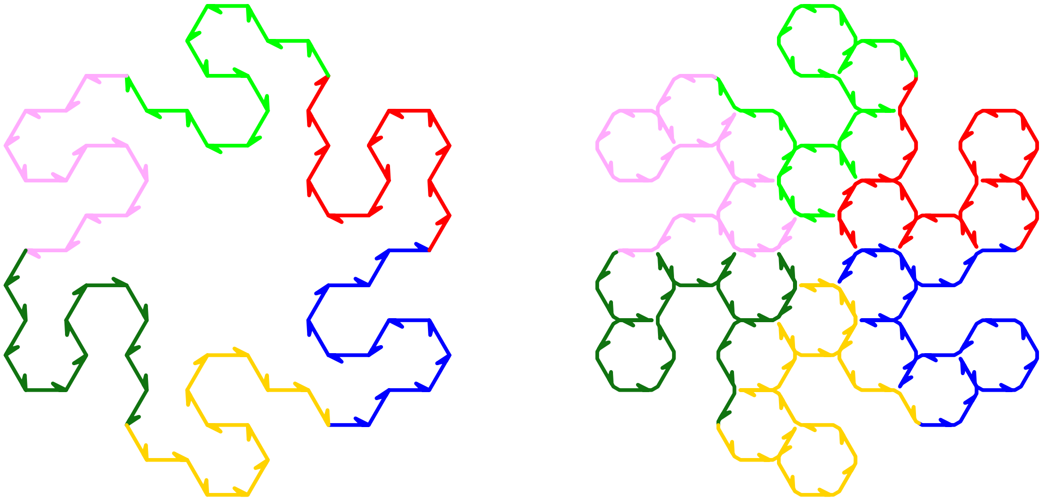

Figure 4.0-B: Border of the CCW tile [A+]^6 (left) and the tile filled with a curve of order 25 (right). First iterates, coloring by orientation of the curves.

A curve of order 25, obtained by filling in the outline of a CCW prototile. The map is

```
A |--> A+A+A!A+A+A+A+A+A-A+A!A+A+A!A+A+A+A+A!A-A-A+A+A-A
```

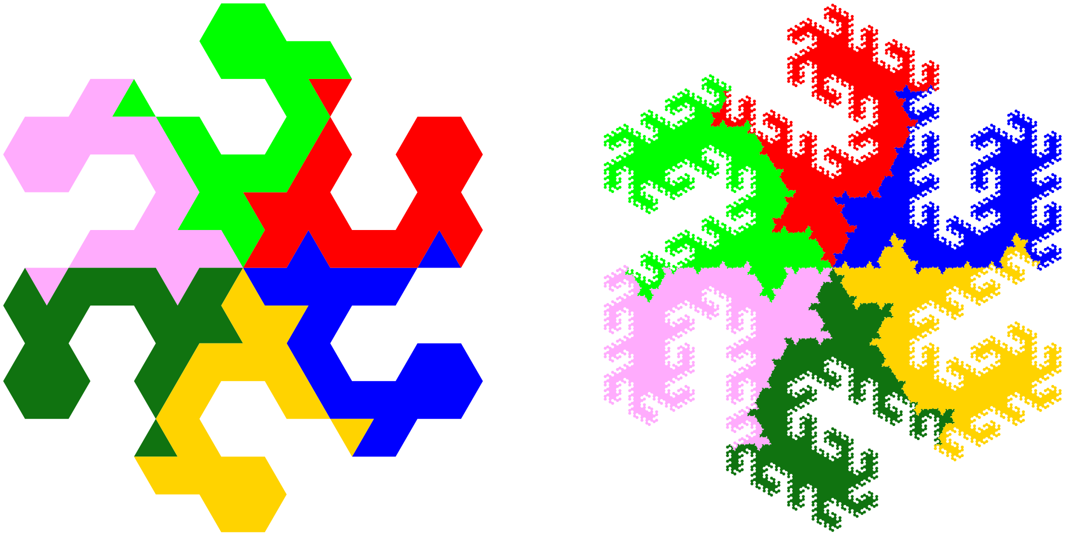

Figure 4.0-C: Area drawings of the first (left) and third (right) iterate of the tile [A+]^6.

The CCW tile.

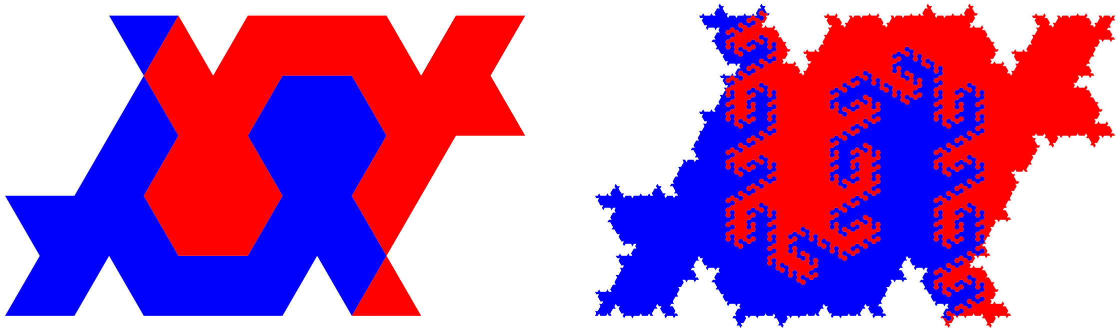

Figure 4.0-D: Area drawings of the first (left) and third (right) iterate of the digon tile [`A!`]^2.

The CW tile.

CW prototile of a curve of order 109

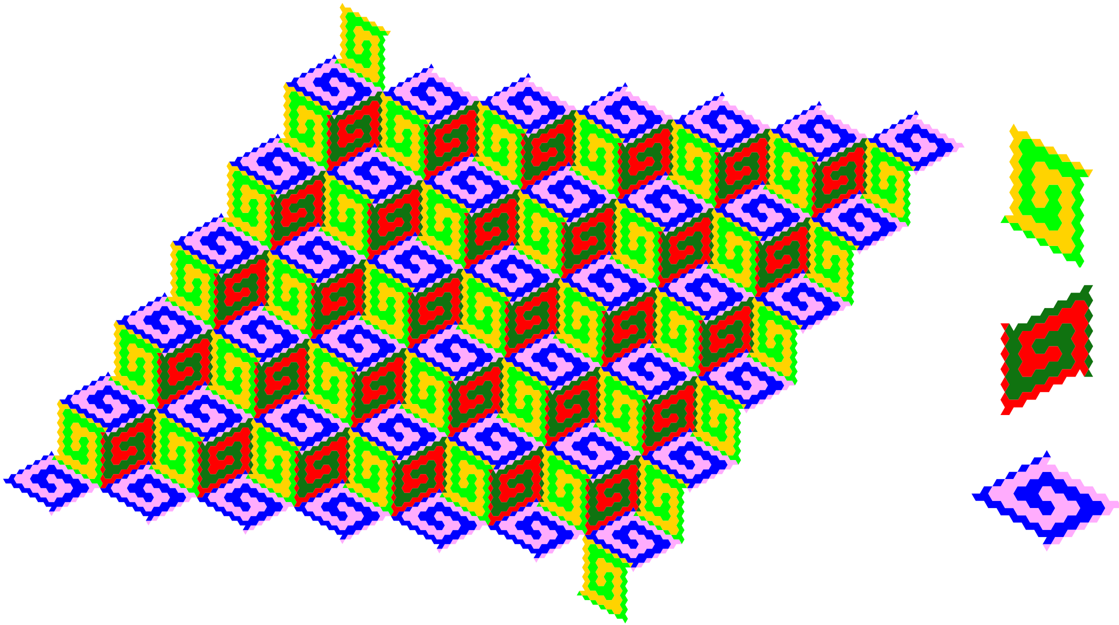

Figure 4.0-E: Second iterate of the digon tile $[\mathtt{A!}]^2$ (left), decomposed into 109 smaller copies of itself (first iterate) appearing in three orientations (right).

CCW prototile of a curve of order 109

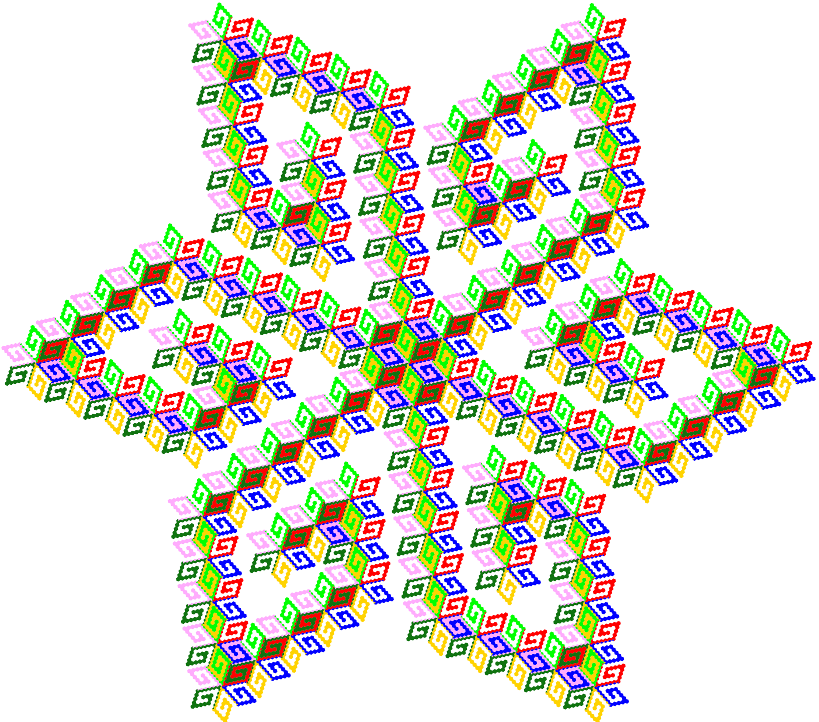

Figure 4.0-F: Second iterate of the CCW prototile [A]^6, decomposed into 109 smaller copies of its first iterate.

Thanks for listening!

Any Questions?

Ask for applications of curve-sets if you want to make me sad.